Exact solution of pulled, directed vesicles with sticky walls in two dimensions

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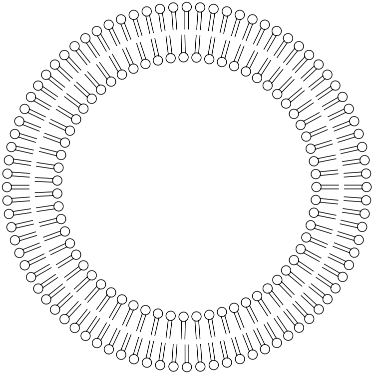
Topic Outline

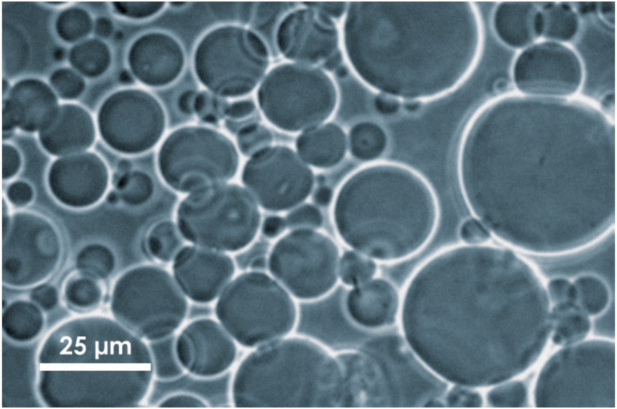
Biological Vesicles

- Vesicles: closed membranes formed of lipid bi-layers



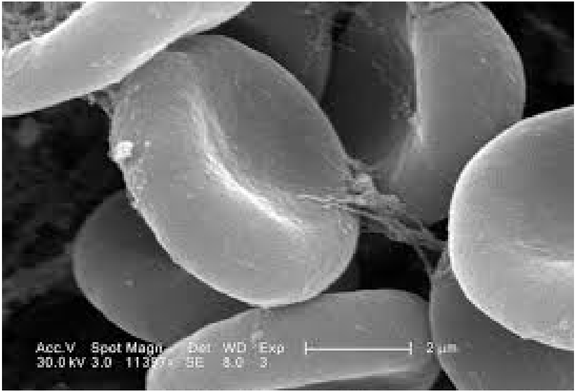
Biological Vesicles

- Vesicles commercially produced by electro-swelling



Biological Vesicles

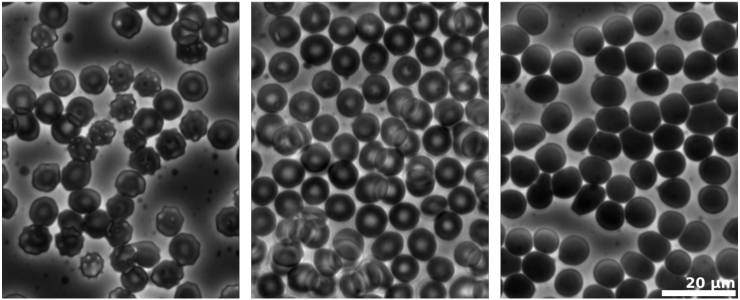
- Red blood cells: lipid bi-layer + spectrin network



Biological Vesicles

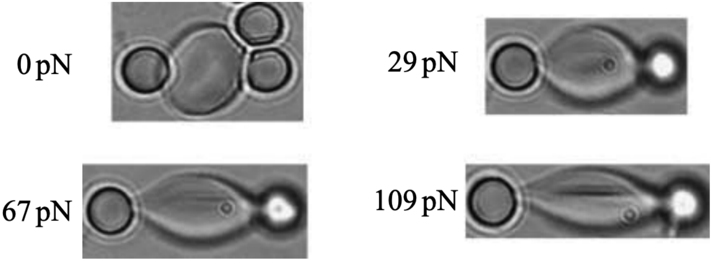
- Red blood cells: effect of osmotic pressure

Hypertonic Isotonic Hypotonic



Biological Vesicles

- Red blood cells pulled with optical tweezer



Modelling Vesicles

We model vesicles as two-dimensional self-avoiding polygons.

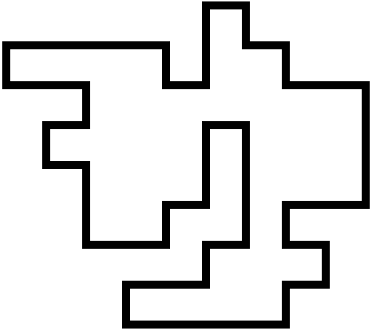


Figure: A self-avoiding polygon of perimeter $2n = 52$ and area $a = 37$.

The Vesicle Generating Function

$$G(q,t) = \sum_{n,a} c_{n,a}\, t^n\, q^a$$

where $c_{n,a}$ is the number of SAP with perimeter $2n$ and area a.

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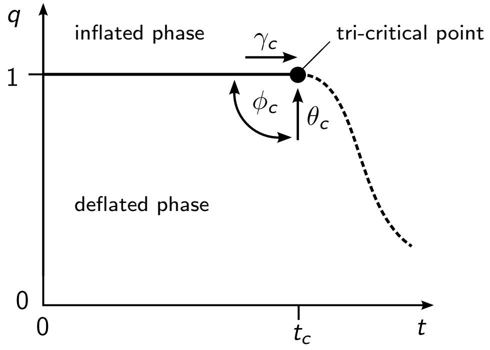

Pulled Directed Sticky Vesicles (PDSV)

- Red blood cell shape $\approx$ preferred direction
- Membrane defects $\approx$ sticky contacts
- Optical tweezer $\approx$ pulling force

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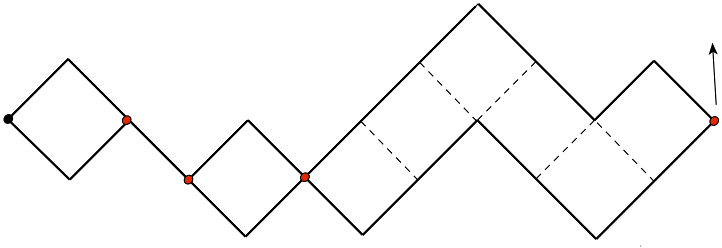
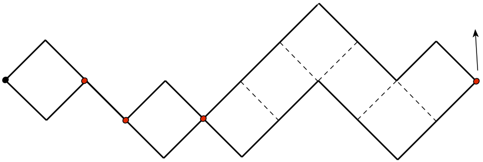


Figure: Two directed walks representing a vesicle with perimeter $2n = 24$ and area $a = 8$, with number of contacts $m = 4$, and indicated pulling force.

The PDSV Generating Function

$$F(c,x,y,q) = \sum_{n_x,n_y,a,m} c_{n_x,n_y,a,m}\, x^{n_x}\, y^{n_y}\, q^a\, c^m$$

where $c_{n_x,n_y,a,m}$ is the number of PDSV with n_x SE steps, n_y NE steps, area a, and m contacts.



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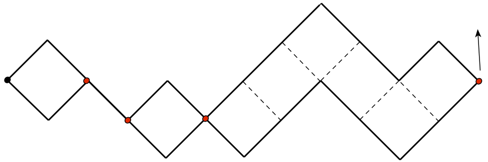


The vertical *endpoint displacement* is given by $h = n_y - n_x$, so introduce the variable $s = e^{-\beta h f}$ conjugate to a *pulling force* f:

$$G(c,s,q,t) = \sum_{m,h,n,a} c_{n_x,n_y,a,m}\, t^{n_x+n_y}\, s^{n_y-n_x}\, q^a\, c^m = F(c,t/s,ts,q)$$

Exact Solution

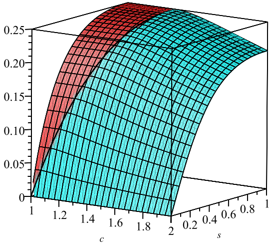
$$F(c,x,y,q) = \cfrac{1}{1 - cx - \cfrac{cy}{1 + y - qx - \cfrac{y}{1 + y - q^2 x - \cfrac{y}{1 + y - q^3 x - \dots}}}}$$

$$= \cfrac{1}{1 - c\left[x + y\cfrac{\sum_{n=0}^{\infty} \frac{(-q^2 x)^n q^{\binom{n}{2}}}{(q;q)_n (qy;q)_n}}{\sum_{n=0}^{\infty} \frac{(-qx)^n q^{\binom{n}{2}}}{(q;q)_n (qy;q)_n}}\right]}$$

where $(t;q)_n = \prod_{k=0}^{n-1} (1 - tq^k)$.

The PDSV Generating Function for $q = 1$

Singularity $t_c(c, s, q = 1)$ of $G(c, s, q = 1, t)$:



- **unbound** and **bound** phases
- **phase transition** at

$$c_s = \frac{(s+1)^2}{s^2+s+1}$$

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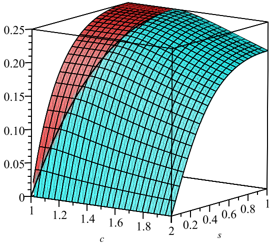

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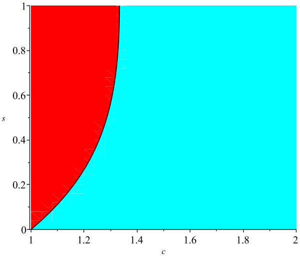
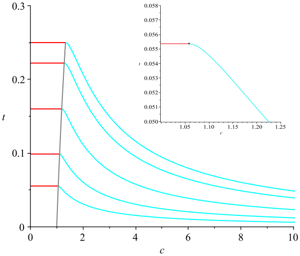
Density of contacts

$$\mathcal{M}(c, s, q = 1) = \begin{cases} 0 \,, & c \leq c_s \\ \sim \frac{2}{c_s^3} \frac{(1+s)^2}{s} (c - c_s) \,, & c > c_s \end{cases}$$

The PDSV Phase Diagram for $q = 1$





- **unbound** and **bound** phases
- The transition looks sharper when decreasing s to zero, but is still smooth
- symmetry between f and $-f$ implies invariance $s \to 1/s$

The PDSV Generating Function for $q \neq 1$

- $q > 1$: The bound phase disappears
  - Configurations with large area $a \sim n^2$ dominate, so $t_c = 0$
- $q < 1$: The unbound phase disappears
  - The density of contacts is positive for any values of s and c

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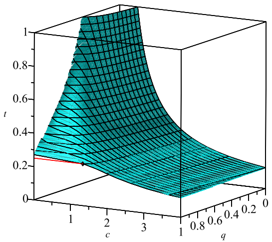

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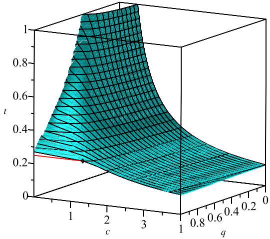

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- No structural difference upon changing s, keep $s = 1$ from now on

Scaling Around the Critical Point

- Near the critical point at $q = 1$ and $c = 4/3$ we find a scaling form

$$c \sim \frac{1}{\frac{3}{4} + 4^{-2/3}\epsilon^{1/3}\frac{\mathrm{Ai}'(4^{1/3}(1-4t_c)\epsilon^{-2/3})}{\mathrm{Ai}(4^{1/3}(1-4t_c)\epsilon^{-2/3})}}$$

with $\epsilon = 1 - q$, where $\mathrm{Ai}(z)$ is the Airy function.

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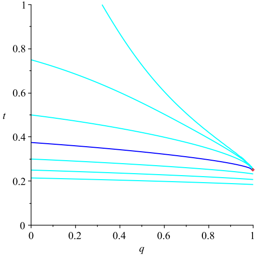
- For $c = 4/3$, and $a_1' = -1.0187\ldots$ the first zero of $\mathrm{Ai}'(z)$:
  - The singularity t_c approaches $1/4$ as
  $$t_c \sim \frac{1}{4} - a_1' 4^{-4/3}\epsilon^{2/3}$$
  - The average area diverges as
  $$\mathcal{A} \sim -a_1' \frac{2^{1/3}}{3}\epsilon^{-1/3}$$
  - The density of contacts vanishes as
  $$\mathcal{M} \sim -\frac{1}{a_1'}\frac{3}{4^{2/3}}\epsilon^{1/3}$$



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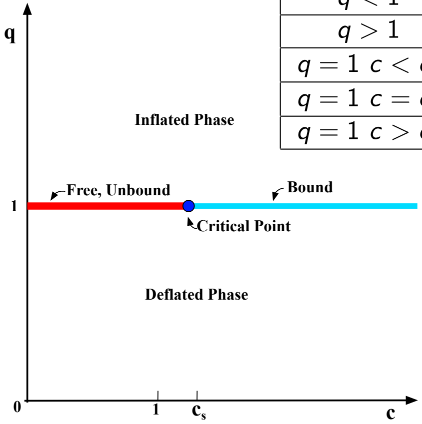

Phase region	Phase	$\langle m \rangle_n$	$\langle a \rangle_n$
$q < 1$	Deflated	n^1	n^1
$q > 1$	Inflated	n^0	n^2
$q = 1$ $c < c_s$	Free, unbound	n^0	$n^{3/2}$
$q = 1$ $c = c_s$	Critical	$n^{1/2}$	$n^{3/2}$
$q = 1$ $c > c_s$	Bound	n^1	n^1

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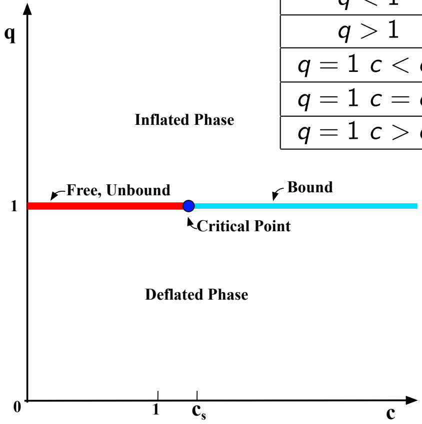

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Any Questions?

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