A tale of two napkins: Interacting Partially Directed Walks

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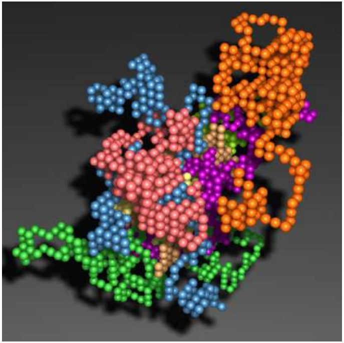
THE UNIVERSITY OF MELBOURNE

Talk outline

1. Some Physics involved in polymers (Motivation)
2. Mathematical Preliminaries
3. Partially directed walks
4. Interacting Partially directed walks
5. Variable flexible Interacting Partially directed walks

Modelling of Polymers in Solution

- Polymers: long chains of monomers
- "Coarse-Graining": beads on a chain
- "Excluded Volume": minimal distance between beads
- Contact with solvent: effective short-range interaction
- Good/bad solvent: repelling/attracting interaction



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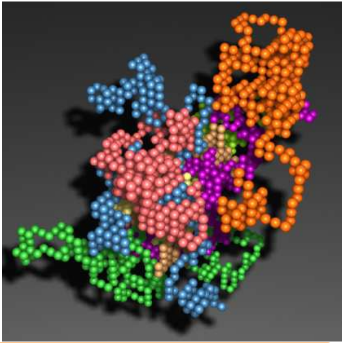
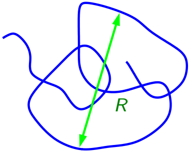


A Model of a Polymer in Solution

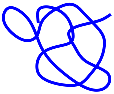
Random Walk + Excluded Volume + Short Range Attraction

A Polymer Phase Transition: Collapse (θ-point)

- *Polymers are often 'Fractal': length* $\mathbf{n}$*, spatial extension* $R \sim n^{\nu}$ *and the mass* $m \propto n \sim R^{d_{fractal}}$ *giving* $\nu = 1/d_{fractal}$.
- *A "Phase transition" occurs as temperature is changed: Polymer Collapse, aka Coil-Globule Transition, aka* Θ*-Point*



$T > T_c$: good solvent swollen phase (coil): $d_{fractal} < d$

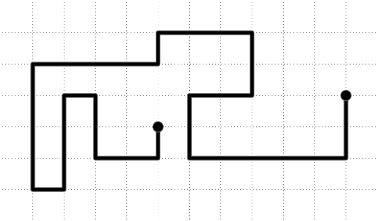


$T = T_c$: Θ-polymer: $d_{fractal} \approx 2$

$T < T_c$: poor solvent — collapsed phase (liquid-like globule): $d_{fractal} = d$

The Canonical Polymer Lattice Model

- *Polymer* $\longrightarrow$ *self-avoiding random walk (SAW)*
- *Physical space* $\longrightarrow$ *regular lattice eg* $\mathbb{Z}^3$ *or* $\mathbb{Z}^2$
- *Sites — beads — monomers*



The Canonical Polymer Lattice Model

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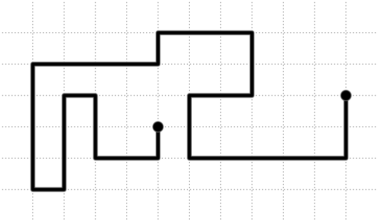
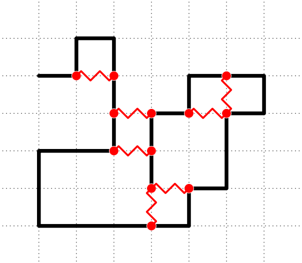


- Introduced by Orr in 1947
- Still no closed form solution for the number of walks of length n
- Can consider limit $n \longrightarrow \infty$ as a critical phenomenon (de Gennes)
  (as well as a phase transition in the thermodynamic limit as temperature is varied).

The Canonical Collapsing Polymer Lattice Model

Interacting Self-Avoiding Walk (ISAW)

- *Start with a SAW and add 'interactions'*
- *Quality of solvent → short-range interaction energy $-J$*
- *Interactions are between (non-consecutive) nearest neighbours*



Variations and extensions

- ISAW even more difficult mathematically
- From work of de Gennes and later by Duplantier and Saleur expect "Tricritical" collapse transition
  - The $n \to \infty$ critical phenomena changes from second order at high temperatures to first order-like at low ones.
  - Thermodynamic limit phase transition is second order
- At low temperatures sometimes can expect β-sheet structures: long folds
- Studies of effects of other forces and physical effects
  - Surface phenomena: polymer adsorption
  - Stiffness: rod-to-coil transition
  - Stretching polymers: micromanipulation experiments
- Search for exactly solvable model ... Start on the square lattice

Generating functions versus partition functions

Partition function

$$Z_n(\text{parameters}) = \sum_{\text{walks of length } n} \text{Boltzmann weight of configuration depending on parameters}$$

and

$$\text{Boltzmann weight} = e^{-\text{Energy of configuration}/k_B T} \qquad \text{eg. } \omega = e^{J/k_B T}$$

where T is the temperature ($\beta = 1/k_B T$), while the generating function is

$$G(z; \text{parameters}) = \sum_{n \geq 0} Z_n(\text{parameters}) \, z^n$$

However, often we evaluate a generating function in terms of composite variables such as

$$x = pz$$

where p is a parameter. This comes about as a natural consequence of the method.

Generating functions versus partition functions II

Let our parameters be $p_j = e^{\beta\epsilon_j}$ where $-\epsilon_j$ is the energy associated with some countable feature of our walk. Let there be $m_j(\varphi_n)$ occurences of this feature in walk configuration φ_n of length n. Then the partition function

$$Z_n(p_1, p_2, \dots) = \sum_{\varphi_n} \prod_j {p_j}^{m_j(\varphi)}$$

and we define a reduced free energy as

$$\kappa(p_1, p_2, \dots) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(p_1, p_2, \dots)$$

Assuming this limit exists the radius of convergence $z_c(p_1, p_2, \dots)$ of the generating function $G(z; p_1, p_2, \dots)$ is related to the free energy as

$$\kappa(p_1, p_2, \dots) = -\log z_c(p_1, p_2, \dots)$$

Phase transition

- Critical phenonemon at $n \to \infty$
- *"Thermal"* phase transition as parameters are varied
- Thermal phase transition implies non-analyticity in free energy and hence z_c as a function of the parameters.
- We expect a collapse phase transition as ω is varied at $\omega = \omega_t = e^{J/k_b T_t}$

Some Exponents

- Generically we expect that for $T \geq T_t$, where T_t is the tricritical collapse θ-temperature

$$Z_n \sim A\, e^{\kappa n}\, n^{\gamma - 1}$$

which implies

$$G(z) \sim \frac{A'}{(z - z_c)^{\gamma}}$$

- Have already met the size exponent ν
- Near the collapse transition we expect

$$\kappa \sim C|T - T_t|^{2-\alpha}$$

This introduces the thermal exponent α.

Some More Exponents

- As we have just introduced, near the collapse transition we expect

$$\kappa \sim C|T - T_t|^{2-\alpha}$$

- Due to the relationship between the free energy and the radius of convergence of the generating function the shape of the curve $z_c(T)$ is related to α.
- The exponent relating the $n \to \infty$ critical phenomena and the thermal phase transition of collapse is the so-called *crossover* exponent ϕ. Eg.

$$R \sim n^{\nu_t} \mathcal{R}((T - T_t)n^{\phi})$$

where $\mathcal{R}$ is a *scaling* function.

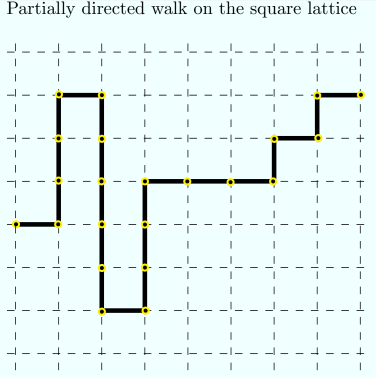
- It can be seen that

$$\phi = 1/(2 - \alpha)$$

Partially directed walks: the beginning

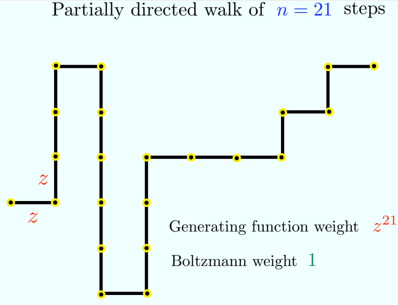
- Soon after Orr's introduction of SAW workers defined restricted sets of SAW so as to analyse SAW (Frisch, Collins and Friedman, 1951)
- Fisher and Sykes (1959) considered partially directed walk configurations so as to provide bounds on SAW numbers
- Temperley (1956) had already considered them as models of magnetic model phase boundaries
- A flurry of activity occurred in the 1980's with Szpilka, Privman, Svrakic, Forgas and Frisch (yes the same one) considering generalisations and adding various parameters

What is a partially directed walk?



- Starting at the origin allow only steps in the $(1, 0)$ (East), $(0, 1)$ (North) and $(0, -1)$ (South) directions

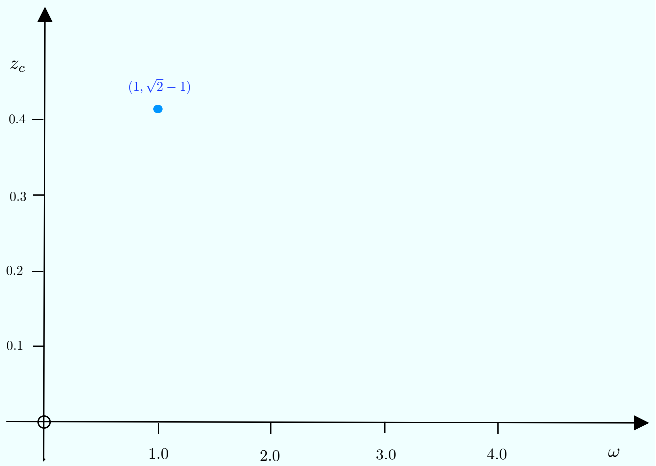
No parameters: our first model



$$G(z) = \frac{1+z}{1-2z-z^2}$$

- The generating function is *rational* with a simple pole ($\gamma = 1$) at $z_c = \sqrt{2} - 1$.

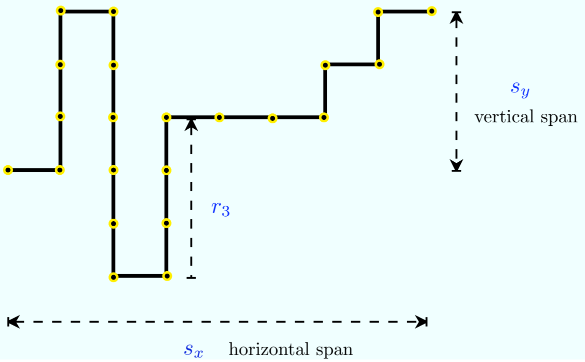
Our first Singularity diagram



Properties of a partially directed walk

Properties of a partially directed walk

n_x horizontal steps and $n_y = \sum_{j=1}^{n_x} |r_j|$ vertical steps



Properties of a partially directed walk

$$s_x = n_x \sim n^{\nu_\parallel}$$

$$s_y \sim n^{\nu_\perp}$$

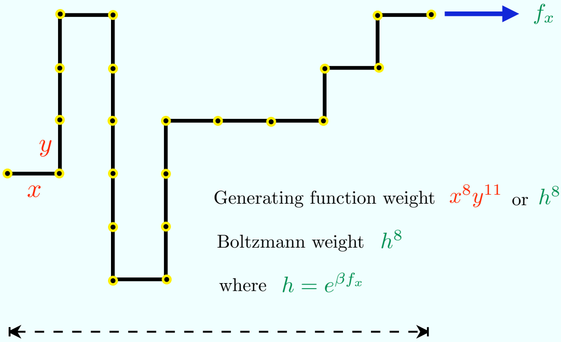
$$\langle |r| \rangle \sim n_r^{\nu}$$

These have been found as $\nu_\parallel = 1$, $\nu_\perp = 1/2$ and $\nu_r = 0$

Counting vertical and horizontal steps separately

Partially directed walk of $n = 21$ steps

including $n_y = 11$ vertical steps, $n_x = 8$ horizontal steps



Generating function weight $x^8 y^{11}$ or $h^8 z^{21}$

Boltzmann weight h^8

where $h = e^{\beta f_x}$

$s_x = n_x = 8$

Counting vertical and horizontal steps separately

The generating stays rational

$$G(x, y) = \frac{1 + y}{1 - y - x - xy}$$

Reinterpreting with a force parameter $h = e^{\beta f_x}$ gives

$$G(z; h) = \frac{1 + z}{1 - z - hz - hz^2}.$$

which has a simple pole at $z_c(h) = \frac{\sqrt{h^2+6h+1}-1-h}{2h}$.

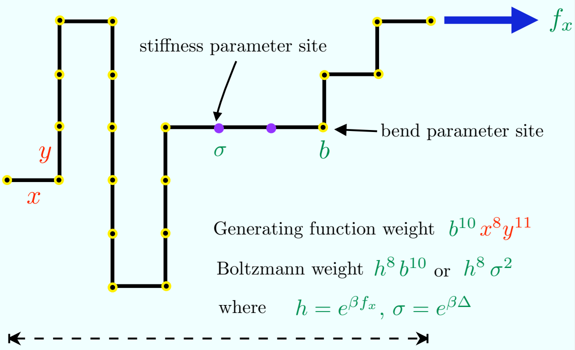
This is an analytic function of $h > 0$ so no finite force produces a phase transition.

Rod to coil Transition

Partially directed walk of $n = 21$ steps

including $n_y = 11$ vertical steps and $k = 10$ bends

equivalently $\ell = \dfrac{2n_x - k - 2}{2} = 2$ stiffness sites



Generating function weight $b^{10} x^8 y^{11}$

Boltzmann weight $h^8 b^{10}$ or $h^8 \sigma^2$

where $h = e^{\beta f_x}$, $\sigma = e^{\beta \Delta}$

$s_x = n_x = 8$

Counting vertical and horizontal steps separately

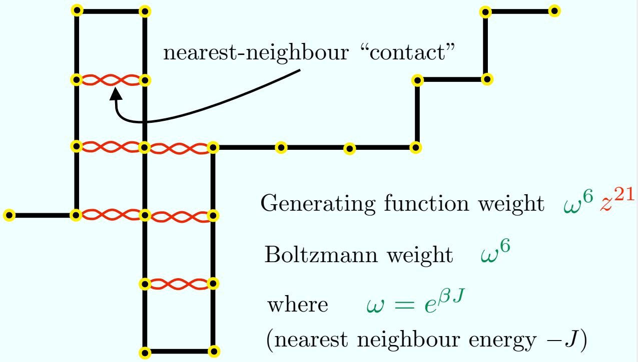
- Privman and Frisch (1988) found the generating function and again it stays rational.

Once again $z_c(b)$ is analytic function of $b > 0$ so no finite bend energy produces a phase transition.
The rod-to-coil "phase transition" is a zero temperature effect.

Interacting partially directed walks

Interacting partially directed walk of $n = 21$ steps

including $m = 6$ nearest neighbour *contacts*



IPDW history

The psychedelic era

- In 1968 and 1970 Zwanzig and Lauritzen considered a more restricted set of walks (partially directed walks that must change vertical direction after each horizontal step).
- They found a collapse transition.
- Their work went essentially unnoticed by the workers on self-avoiding walks.

The Thatcher Years

- During the 80's there was lots of work on magnetic/liquid phase boundaries including so-called Solid-on-solid models (aka Partially directed walks in 2 dimensions)
- Also work on related polymer adsorption transition in late 80s/early 90s

IPDW history

- In 1989 one generalisations of the adsorption problem was considered by Veal, Yeomans (Oxford) and Jug.
- They considered partially directed walks above a sticky wall but also with nearest neighbour attraction.
- They numerically analysed the transfer matrices and found an accurate value of the tricritical point without any surface attraction at

$$(\omega, z) = (3.382976, 0.2955977)$$

- ALO moves to Oxford and becomes interested in interfacial phenomena like depinning and wetting transitions
- One thing leads another and IPDW is considered

Solution on the special curve $z = 1/\omega$

Binder, Owczarek, Veal and Yeomans (1990) used the transfer matrix method to examine the model along the curve

$$z = 1/\omega$$

They found a singular point at the solution of a cubic giving

$$\omega_t^{-1} = \frac{1}{9}\left[(17+3\sqrt{33})^{1/3} + (17-3\sqrt{33})^{1/3} - 1\right]^2 = 3.3829757\ldots$$

They argued on physical grounds that this must be the tricritical point. Further that for $\omega > \omega_t$ one has $z_c(\omega) = 1/\omega$.

IPDW history

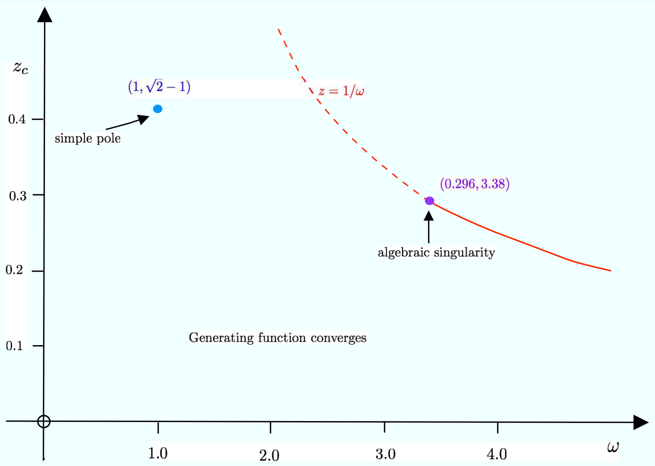
- Using generating function methods one can easily find that

$$1 + G(z; \omega = 1/z) = \sqrt{\frac{1-z}{1-3z-z^2-z^3}}$$

- For $\omega > 3.382976\ldots$ the generating function is finite
- Along this curve the singularity at $(3.382976\ldots, 0.2955977\ldots)$ is algebraic: a divergent square root singularity to be precise.
- We define

$$\gamma_t^{tangential} = 1/2$$

Singularity diagram 2



The other side of the world

- Whittington (Toronto) and Guttmann (Melbourne) continue their quests for solution of SAW problems
- Brak moves from Oxford to Melbourne and becomes interested in SAWs and Polymers
- Brak, Guttmann and Whittington investigate IPDW

Temperley method

Now Brak and Guttmann had uncovered the method used by Temperley in 1956

- Consider the generating function for walks with last vertical segment of length r
- Find a recurrence for these by adding an extra column of walk
- Use a q-series Ansatz

Solution

Let us define the variable

$$q = \omega z$$

so the special curve becomes $q = 1$.
Solving the recurrence gives

$$1 + G(z;\omega) = \frac{z^2(z-1)g_0^{(1)}}{z^2(1+\omega+z-\omega z)g_0^{(1)} - 2zg_1^{(1)}}.$$

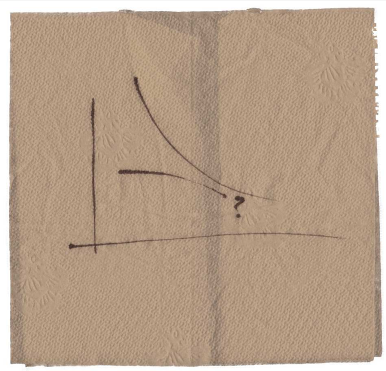
where

$$g_r^{(1)} = z^r + z^r \sum_{m=1}^{\infty} \frac{(z-q)^m q^{m(m+1)/2}}{\prod_{k=1}^{m}(q^k-\omega)\prod_{k=1}^{m}(q^k-1)} q^{mr}.$$

q-Bessel

- The functions involved were later indentifies as q-Bessel functions.
- Each $g_r^{(1)}$ converges for any $q < 1$
- So the singularities of $G(z;\omega)$ may be at $q^k = 1$ and $q^k = \omega$ for all natural numbers k
- Singularites at $q^k = \omega$ for all natural numbers k actually occur
- This implies essential singularities on approaching $z = 1/\omega$ from below.
- There may also be a simple pole when the denominator is zero: this certainly happens when $\omega = 1$ and for some $\omega > 1$.
- These singularities should meet at the tricritical point.
- But how does one **prove** it!
- So they went to lunch ...

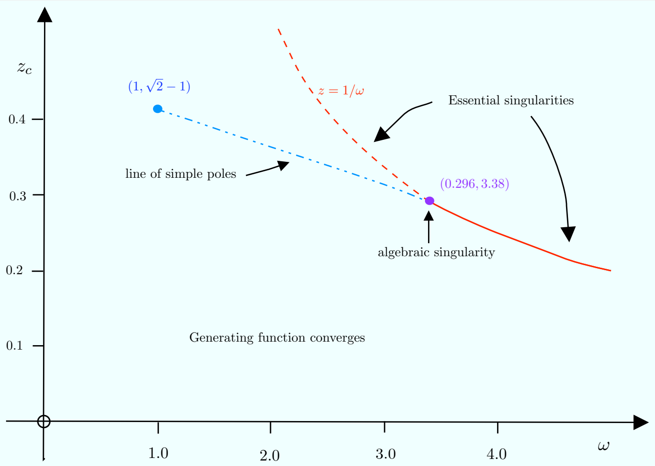
Napkin 1



The proof

- Can find a functional equation for the functions $g_r^{(1)}$ which lead to a continued fraction expansion
- Use theory of continued fractions
- Worpitzky's theorem

Final singularity diagram for IPDW



But is it collapse?

Prellberg moves to Melbourne and then Owczarek moves to Melbourne

- Prellberg, Owczarek, Brak and Guttmann 1993 produced series using the recurrences for walks of length up to several thousand.
- They numerically verified the scaling predictions of Owczarek, Prellberg and Brak 1993

So what did Owczarek, Prellberg and Brak 1993 do?

OPB 1993

- Made predictions for the scaling of properties above, below and around the collapse point.
- Solved a semi-continuous variant of IPDW and showed explicitly the tricritical scaling around ω_t
- Generalised the solution of Brak, Guttmann, Whittington to distinguish horizontal to vertical steps
- Use perturbation expansions (non-rigorous) to demonstrate tricritical scaling in the discrete IPDW model
- Showed how the generating function of the semi-continuous model was a simple limit of the discrete model
- Connect the work of Zwanzig and Lauritzen and IPDW
- Relate SOS model of magnetic model interfaces and IPDW generating functions
- In an appendix solve a generalisation that distinguishes positive and negative vertical steps

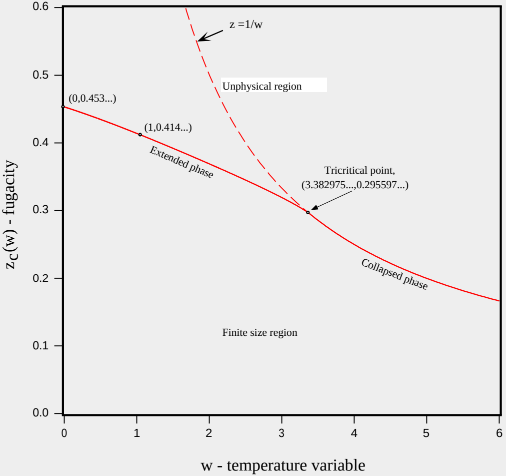
Scaling 1

The following were found exactly for the semi-continuous model and verified for the discrete IPDW

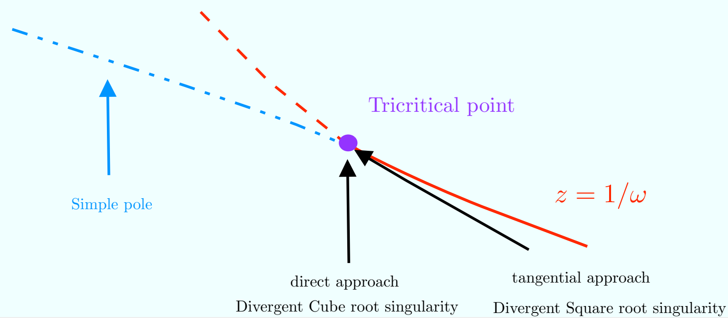
- At high temperature $\omega < \omega_t$ we have $\gamma = 1$, $\nu_{\|} = 1$, $\nu_{\perp} = 1/2$ and $\nu_r = 0$
- At the collapse point precisely $\omega = \omega_t$ we have $\gamma \equiv \gamma_t^{direct} = 1/3$, $\nu_{\|} = 2/3$, $\nu_{\perp} = 1/3$ and $\nu_r = 1/3$
- For low temperatures we have $\nu_{\|} = 1/2$, $\nu_{\perp} = 1/4$ and $\nu_r = 1/2$

Later work by Owczarek (1993) shows the low temperature scaling of the partition function does not follow the usual form

Singularity diagram: calculated from continued fractions

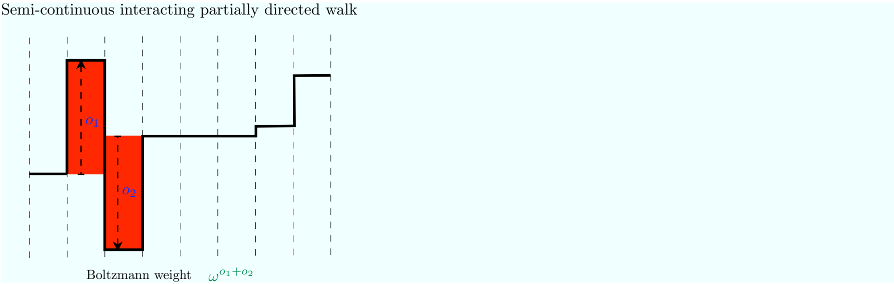


Tricritical point scaling region



Semicontinuous IPDW

- A variant is to allow the vertical sections of the walk be of any *Real* length rather than discrete
- Naturally horizontal and vertical "steps" are distinguished



Solution of the semi-continuous model

The solution can be written in terms of Bessel functions:

$$1 + G_{semi-cont}(x, y; \omega) = \sigma^{-1} \frac{J_\lambda(\sigma\lambda)}{J'_\lambda(\sigma\lambda)}$$

where $\sigma = \left(\frac{4x}{\beta J}\right)^{1/2}$ and $\lambda = \frac{\beta J}{\tau - \beta J}$ while $\omega = e^{\beta J}$ and $y = e^{-\tau}$
Near the tricritical point we have

$$G_{semi-cont}(x = 1, y(\lambda); \omega(\zeta)) \approx -\left(\frac{\zeta}{1-\sigma^2}\right)^{1/2} \frac{\mathrm{Ai}(\lambda^{2/3}\zeta)}{\mathrm{Ai}'(\lambda^{2/3}\zeta)} \lambda^{1/3}$$

Tricritical scaling

This gave us verification of the general scaling ansatz

$$G(z;\omega) \sim \frac{1}{(z-z_t)^{\gamma_t}} \mathcal{S}\left(\frac{\omega-\omega_t}{(z_t-z)^{\phi}}\right)$$

where $z_t \equiv z_c(\omega_t)$ with $\gamma_t \equiv \gamma_t^{direct} = 1/3$ and the crossover exponent $\phi = 2/3$.

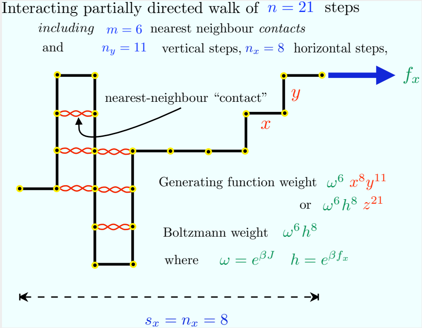
This is consistent with our calculated value of $1/2$ for $\gamma_t^{tangential}$ as we must have

$$\gamma_t^{tangential} = \gamma_t^{direct}/\phi$$

It also follows that the thermal exponent $\alpha = 2 - \frac{1}{\phi}$=$1/2$.

IPDW differentiating horizontal and vertical steps

Another variation solved was distinguishing horizontal and vertical steps with counting variables x and y: similar q series solution



Horizontal force interpretation not realised at the time however!

The final nail

- In 1995 Prellberg rigorously proved uniform asymptotic results for q-Bessel functions in the context of the *Bar graph* polygon model.
- However the functions involved in IPDW are related
- So we left it at that .... after all, all the results were in the literature, though not in one place.

Stretching polymer

- With the advent of instruments in the early naughties that can micromanipulate single polymer molecules there was a renewed interest in the behaviour of polymers when placed under a stress
- Rosa, Marenduzzo, Maritan, and Seno (2003) cleverly reinterpreted IPDW distinguishing horizontal and vertical steps as a model of stretching (horizontally) collapsing polymers.
- This can be done simply by substituting

$$x = hz, \qquad y = z \qquad \text{with } h = e^{\beta f_x}$$

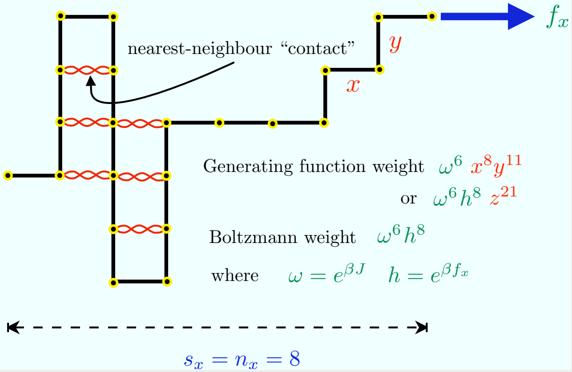
- They plotted critical force-temperature curves and conjectured various exponents.

IPDW differentiating horizontal and vertical steps

Interacting partially directed walk of $n = 21$ steps

including $m = 6$ nearest neighbour *contacts*

and $n_y = 11$ vertical steps, $n_x = 8$ horizontal steps,



Adding stiffness to collapsing polymers

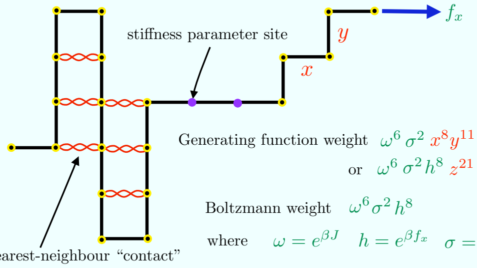
- For ISAW it was seen by Bastolla and Grassberger (1997) that adding stiffness can change the collapse transition to first order
- Different in effect in different dimensions and needs sufficient stiffness
- Zhou, Zhou, Zhong-Can Ou-Yang, and Kumar (*Phys. Rev. Lett.* (2006) studied
  "Collapse Transition of Two-Dimensional Flexible and Semiflexible Polymers"
  - Actually considered IPDW modified
  - Used analytic approximation and Monte Carlo
  - Conjectured that the collapse transition becomes first order with any stiffness added

Interacting partially directed walk of $n = 21$

including $m = 6$ nearest neighbour *contacts*,

$n_y = 11$ vertical steps, $n_x = 8$ horizontal steps,

$\ell = 2$ stiffness sites



Generating function weight $\omega^6 \sigma^2 x^8 y^{11}$

or $\omega^6 \sigma^2 h^8 z^{21}$

Boltzmann weight $\omega^6 \sigma^2 h^8$

where $\omega = e^{\beta J}$ $h = e^{\beta f_x}$ $\sigma = e^{\beta \Delta}$

$s_x = n_x = 8$

Solution

Owczarek and Prellberg, *J. Stat. Mech.: Theor. Exp.*, P11010:1-14, 2007
"Exact Solution of Semi-Flexible and Super-Flexible Interacting Partially Directed Walks"
The solution for the generating was found using the Temperley-like methodology:

$$1+G(z;h,\sigma,\omega)=\frac{(1-\omega)}{2}\left[\mathcal{H}(z,z\omega,hz^2(\omega-1))-\left(\frac{(1+\omega)}{2}+\frac{(1-\omega)}{2}\frac{hz}{1-hz(\sigma-1)}\right)\right]^{-1}$$

where

$$\mathcal{H}(y,q,t)=\frac{H(y,q,qt)}{H(y,q,t)}\quad\text{and}\quad\mathcal{H}(y,1,t)=\frac{1}{2y}\left[1+y-t-\sqrt{(1+y-t)^2-4y}\right]$$

with

$$H(y,q,t)=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(-t)^n}{(y;q)_n(q;q)_n}$$

Solving along $q = 1$

Let $u = \omega/(\omega - h(\sigma - 1))$, then along the curve $q = 1$ the generating function has an algebraic singularity at

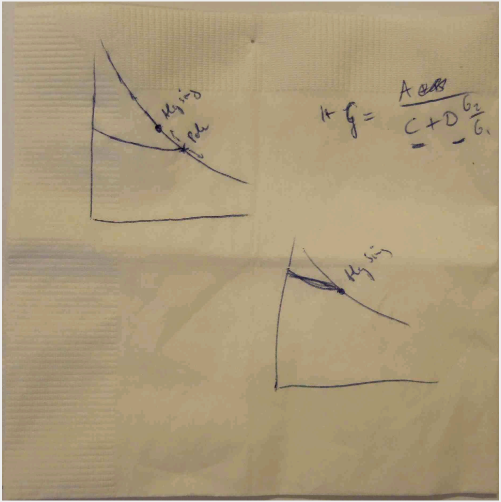
$$\omega_a(h) = \left(\frac{\omega + h}{\omega - h}\right)^2$$

and for $u > 1$ a simple pole at

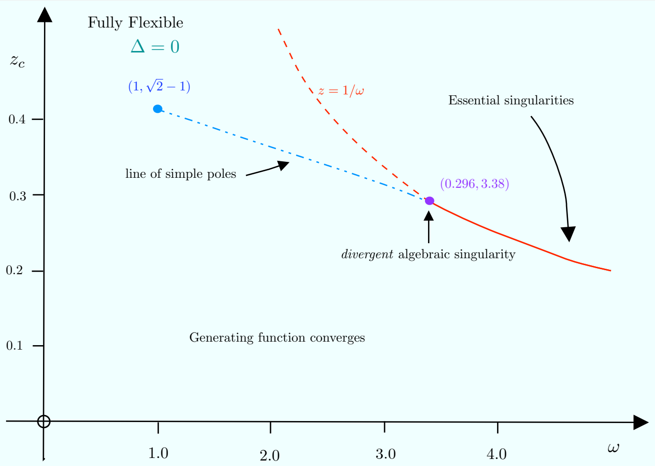
$$\omega_p(h, u) = \frac{(\omega + uh)(\omega + 2h - uh)}{(\omega - uh)(\omega - 2h + uh)}$$

The singularites coincide for $u = 1$, ie. $\sigma = 1$.
So, minor change to the solution: this changes the singularity structure it seems ... lunch

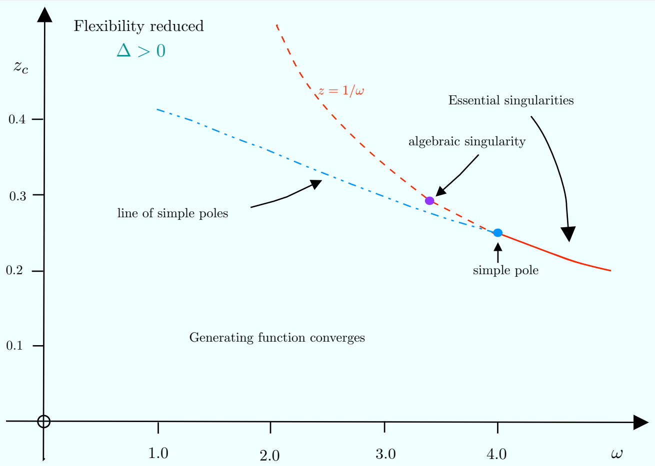
Napkin 2



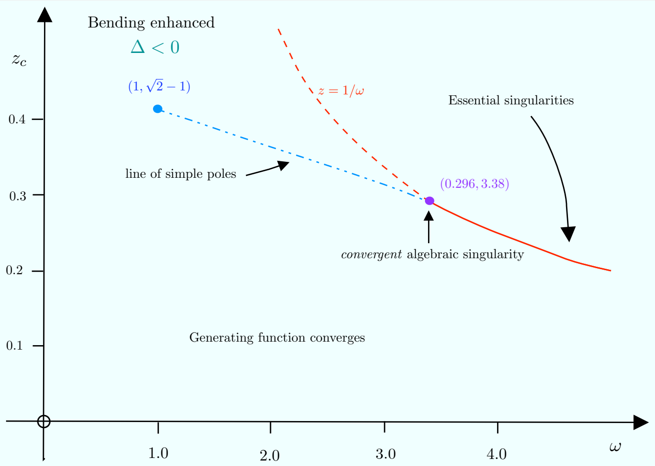
Singularity diagram: no stiffness



Singularity diagram: adding positive stiffness



Singularity diagram: encouraging bends



Exponents

Fully flexible case ($\Delta = 0$): we have a second-order collapse transition with

$$\gamma_t^{tangential} = \frac{1}{2} \qquad \gamma_t^{direct} = \frac{1}{3} \qquad \phi = \frac{2}{3} \,,$$

Super-flexible case ($\Delta < 0$): we have a second-order collapse transition with

$$\gamma_t^{tangential} = -\frac{1}{2} \qquad \gamma_t^{direct} = -\frac{1}{3} \qquad \phi = \frac{2}{3} \,,$$

Semi-flexible case ($\Delta > 0$): first order transition.

Scaling region 1

Using Lemma 4.3 from Prellberg (1995), a result completely analogous to Theorem 5.3 in Prellberg (1995) can be obtained for $\mathcal{H}(y, q, t)$, i.e. an asymptotic expansion in $q = 1 - \epsilon$ uniformly valid for all values of t and y, which reads

$$\mathcal{H}(y, 1-\epsilon, t) = \frac{1}{2y}\left[1 + y - t - \left(-\frac{\mathrm{Ai}'(\alpha\epsilon^{-2/3})}{\alpha^{1/2}\epsilon^{-1/3}\,\mathrm{Ai}(\alpha\epsilon^{-2/3})}\right)\sqrt{(1+y-t)^2 - 4y}\right](1+O(\epsilon)) .$$

Here, $\alpha = \alpha(y, t)$ is a function of y and t which is known exactly.
Near the tricritical point

$$\alpha(y, t) \sim \left(\frac{4}{1-(t-y)^2}\right)^{4/3} \frac{(1+y-t)^2 - 4y}{4}$$

for small $(1 + y - t)^2 - 4y$.

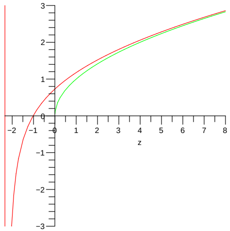
Scaling region 2

$$\mathcal{H}(y, 1-\epsilon, t) \sim \frac{1}{2y}\left[1 + y - t + \epsilon^{1/3}\frac{\mathrm{Ai}'(\alpha\epsilon^{-2/3})}{\mathrm{Ai}(\alpha\epsilon^{-2/3})}\frac{(1-(t-y)^2)^{2/3}}{2^{1/3}}\right] .$$

The behaviour of this expression is determined by the function $f(z) = -\mathrm{Ai}'(z)/\mathrm{Ai}(z)$

The function $f(z) = -\operatorname{Ai}'(z)/\operatorname{Ai}(z)$

- The large-z asymptotics allows for matching for $\epsilon \to 0$ and positive α.
- For negative α, the argument of f is negative. As $f(z)$ has a simple pole at $z = -2.3381\ldots$, for any fixed $\alpha < 0$ we have a pole at a finite value of ϵ.
- As α tends to zero, the locus of this pole scales as $\epsilon^{2/3}$.



The function $\sqrt{z}$ is plotted for comparison.

What next?

Interacting partially directed walk of $n = 21$

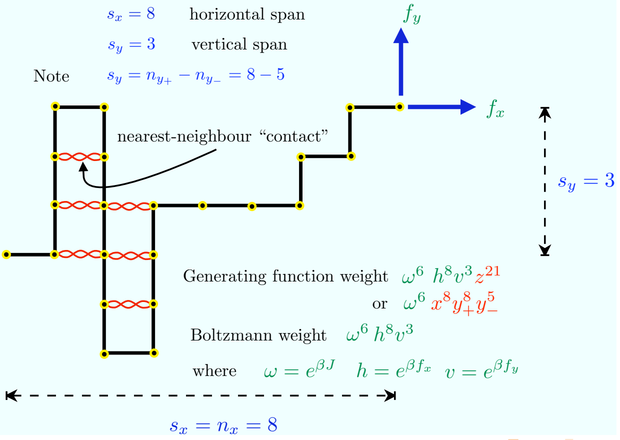
including $m = 6$ nearest neighbour *contacts*

and $n_y = 11$ vertical steps, $n_x = 8$ horizontal steps,

$s_x = 8$ horizontal span

$s_y = 3$ vertical span

Note $s_y = n_{y_+} - n_{y_-} = 8 - 5$



Generating function weight $\omega^6 \; h^8 v^3 z^{21}$

or $\omega^6 \; x^8 y_+^8 y_-^5$

Boltzmann weight $\omega^6 \; h^8 v^3$

where $\omega = e^{\beta J}$ $\quad h = e^{\beta f_x}$ $\quad v = e^{\beta f_y}$

The solution

The solution can be found in (you guessed it) Owczarek, Prellberg and Brak (1993) as

$$1 + G(x, y_+, y_-, \omega) = \frac{1-\omega}{2} \left\{ \mathcal{H}(x, y_+, y_-, \omega) - \left(\frac{1+\omega}{2} + \frac{1-\omega}{2} x\right) \right\}^{-1}$$

where

$$\mathcal{H}(x, y_+, y_-, \omega) \quad = \quad \frac{(A_0^+ + B_0^+)(A_1^+ - B_1^+) - (A_0^- + B_0^-)(A_1^- - B_1^-)}{(A_0^+ + B_0^+)(A_0^+ - B_0^+) - (A_0^- + B_0^-)(A_0^- - B_0^-)} \, .$$

and

$$\begin{aligned} A_r^{\pm} &= \sum_{m=0}^{\infty} \frac{x^{2m}(\omega-1)^{2m}(q_+q_-)^{m(m+r)} {q_\pm}^{m}}{\prod_{k=1}^{m} P[(q_+q_-)^{k-1}q_\pm] P[(q_+q_-)^k]} \\ B_r^{\pm} &= \sum_{m=0}^{\infty} \frac{x^{2m+1}(\omega-1)^{2m+1}(q_+q_-)^{m(m+r)} {q_\pm}^{r+m+1}}{P[(q_+q_-)^m q_\pm] \prod_{k=1}^{m} P[(q_+q_-)^{k-1}q_\pm] P[(q_+q_-)^k]} . \end{aligned}$$

and $P[l] = (l-1)(l-\omega)$ Note that $q_+q_- = q^2$, and given that all the parameters are positive we have $q = \sqrt{q_+q_-}$. But what can we say about the singularity structure, phase transition, scaling etc ...

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- The literature contains the occasional gem (ie Temperley, Zwanzig/Lauritzen etc)
- Always try to reinterpret your own work
- Lunch is a useful scientific tool