Directed Compact Percolation near a damp wall

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THE UNIVERSITY OF MELBOURNE

AUSTRALIAN RESEARCH COUNCIL
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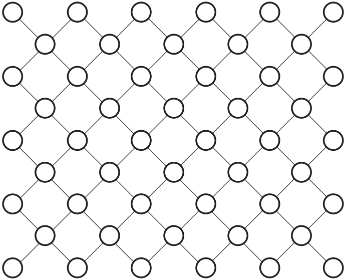
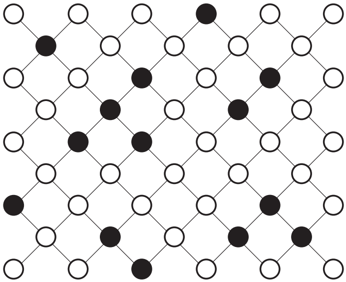
A brief introduction to percolation theory

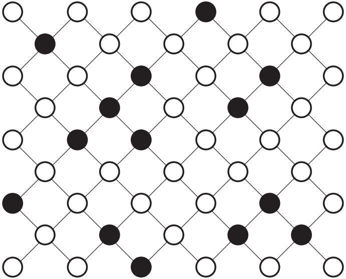


A brief introduction to percolation theory



- each site occupied (wet) with probability p

A brief introduction to percolation theory



- each site occupied (wet) with probability p
- unoccupied (dry) with probability $q = 1 - p$

A brief introduction to percolation theory

- cluster size distribution
- critical behaviour – phase transition at $p = p_c$
- for $p \leq p_c$, all clusters are finite
- for $p > p_c$, there exists an infinite cluster
- probability of a given site being part of an infinite cluster $=$ percolation probability $P(p)$

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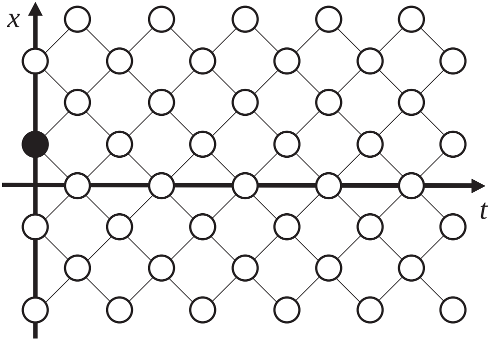
$$P(p) \sim (p - p_c)^{\beta}, \quad p > p_c$$

- $\beta =$ critical exponent for percolation probability

Directed Percolation



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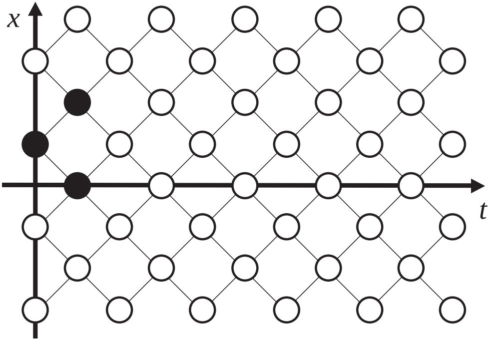

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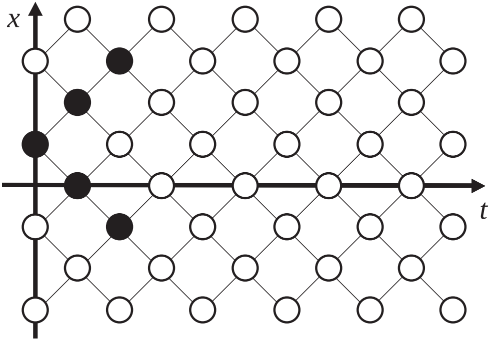

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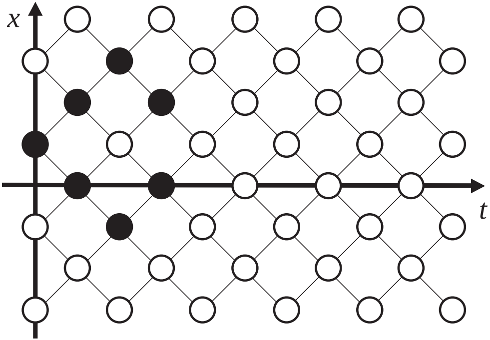

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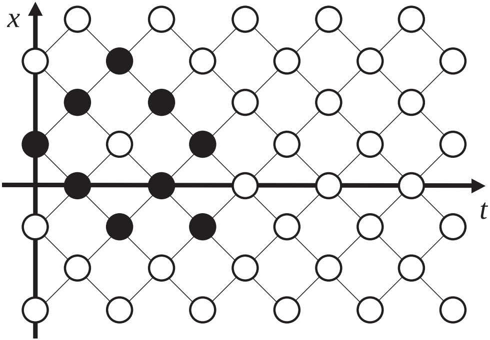

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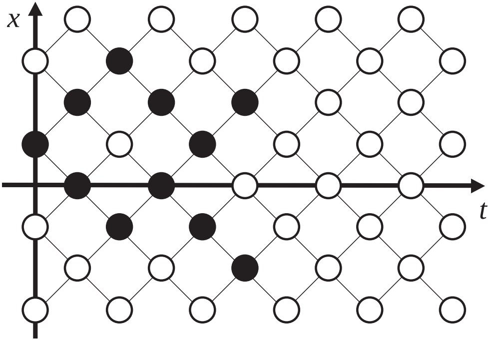

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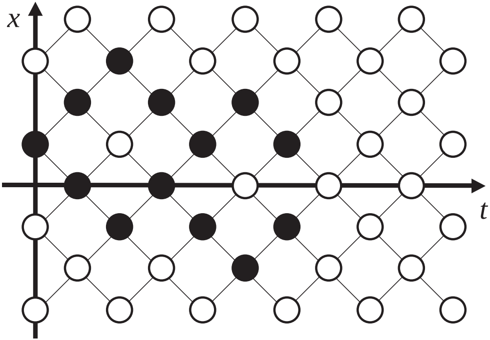

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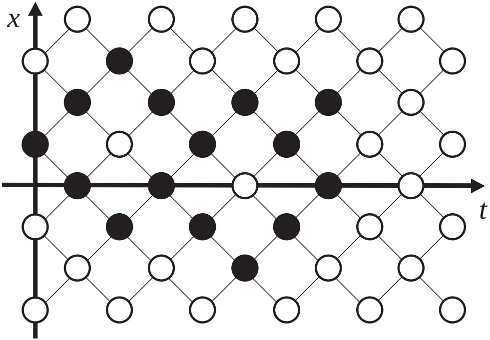

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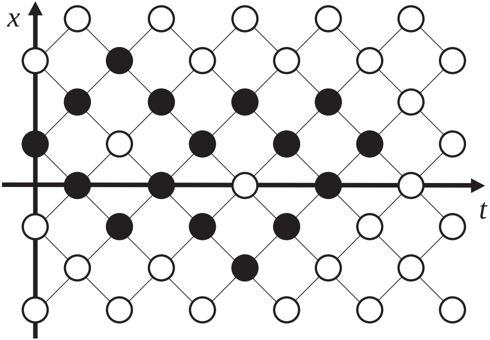

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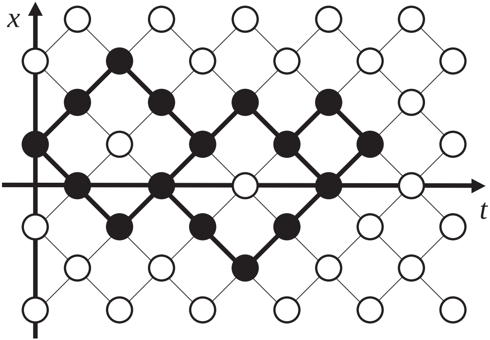

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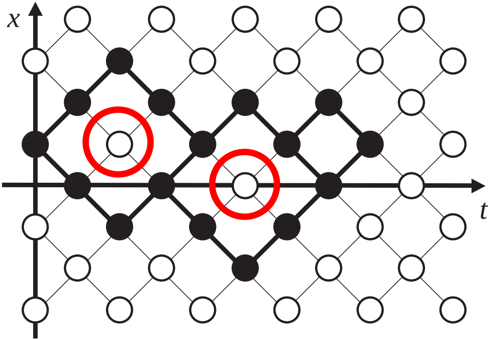
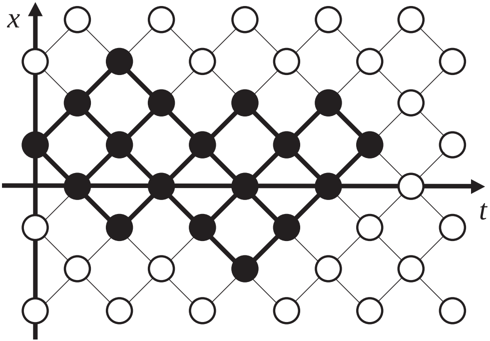


Directed Compact Percolation



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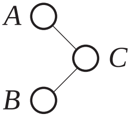

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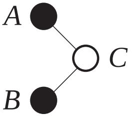

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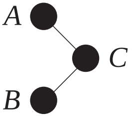
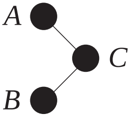
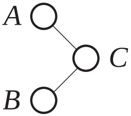


Pr(C occupied): 1

Directed Compact Percolation





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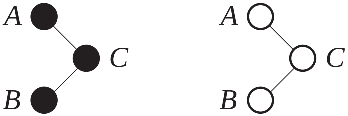
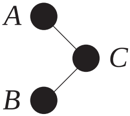
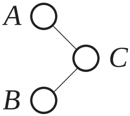


Pr(C occupied): 1 0

Directed Compact Percolation





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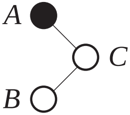

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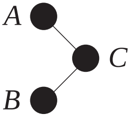

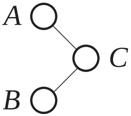

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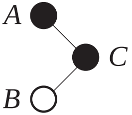
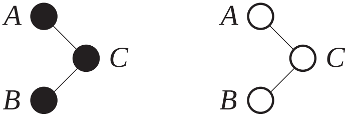
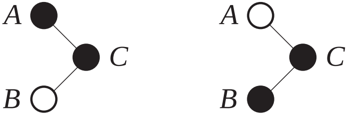


Pr(C occupied): p

Directed Compact Percolation

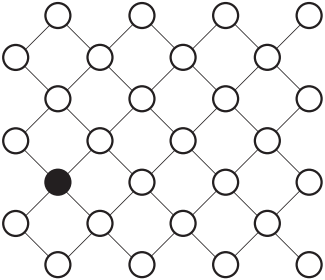


Pr(C occupied): 1 0



Pr(C occupied): p p

Directed Compact Percolation



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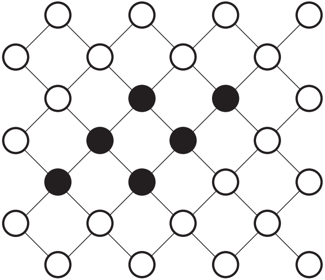

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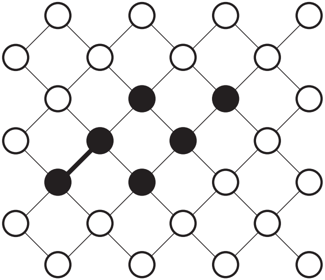
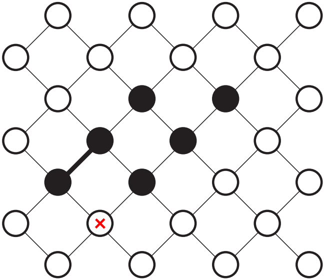


p

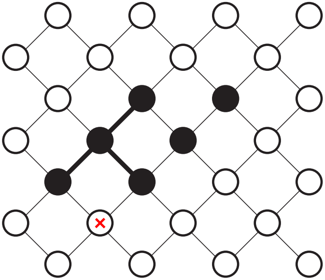
Directed Compact Percolation



pq

$q = 1 - p$

Directed Compact Percolation



$pq \quad p^2$

$q = 1 - p$

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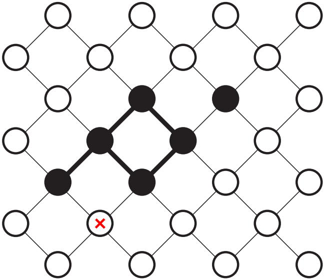

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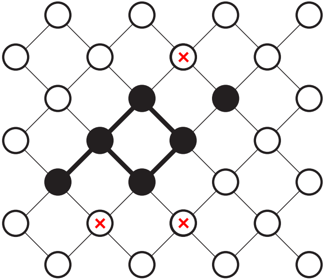

$pq \quad p^2 \quad q^2$

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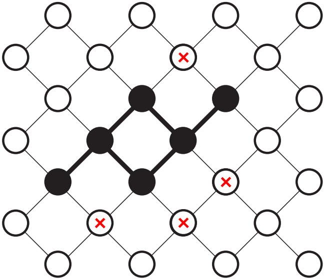

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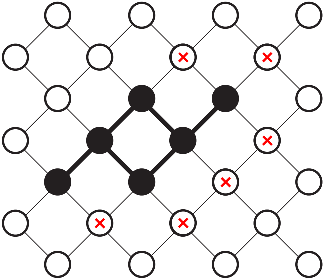

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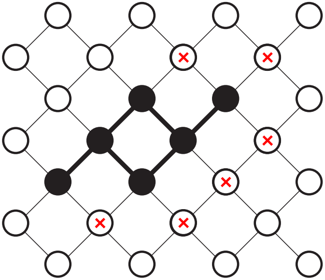


$pq \quad p^2 \quad q^2 \quad pq \quad q^2$

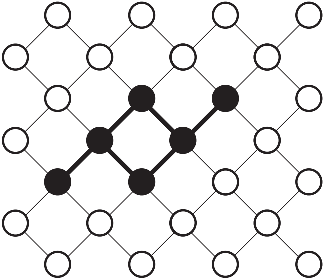
probability of cluster $= p^4 q^6, \qquad q = 1 - p$

Directed Compact Percolation

$$Q(p) = \text{sum of probabilities of finite clusters}$$

$$P(p) = 1 - Q(p)$$
$$= \text{percolation probability}$$

Directed Compact Percolation



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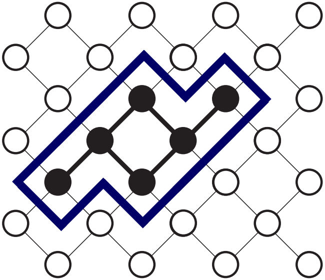

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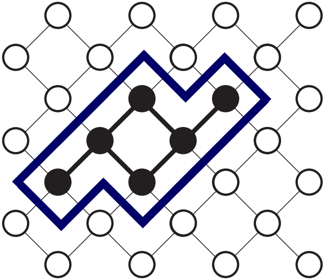
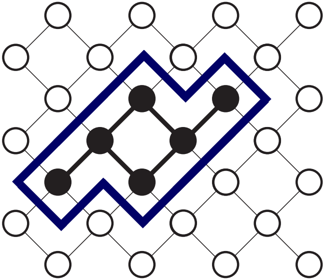


z = weighting on each step of the walk

$G(z)$ = generating function for staircase polygons

Directed Compact Percolation



z = weighting on each step of the walk

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$$Q(p) = \frac{1}{p^2} G(pq)$$

Directed Compact Percolation

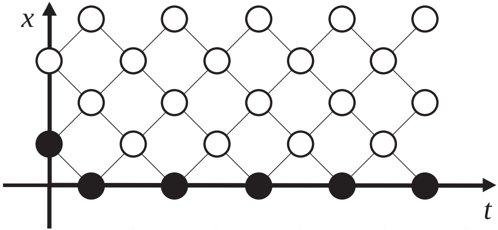
$$G(z) = \frac{1 - 2z - \sqrt{1-4z}}{2}$$

$$Q(p) = \frac{1 - 2p(1-p) - |2p-1|}{2p^2}$$

$$P(p) = \begin{cases} 0, & p \leq \frac{1}{2} \\ \frac{2p-1}{p^2}, & p > \frac{1}{2} \end{cases}$$

critical exponent $\beta = 1$

Directed Compact Percolation near a wet wall



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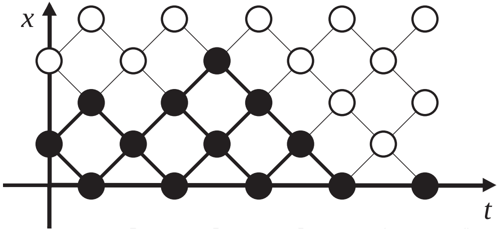

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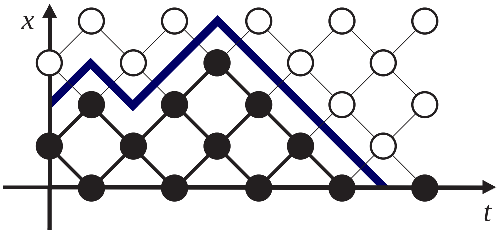
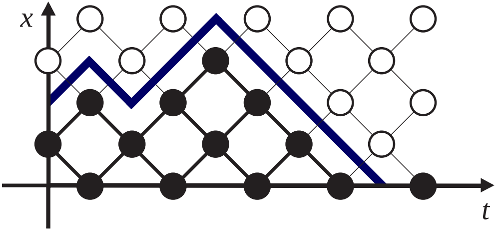


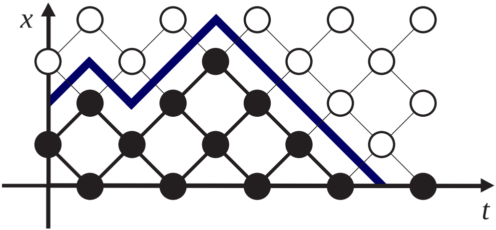
- relate to a directed walk – Dyck paths

Directed Compact Percolation near a wet wall



- relate to a directed walk – Dyck paths
- weight walk with κ each time it touches $x = 1$
- $G(z, \kappa) =$ generating function for walks

Directed Compact Percolation near a wet wall



- relate to a directed walk – Dyck paths
- weight walk with κ each time it touches $x = 1$
- $G(z, \kappa) =$ generating function for walks
- $Q(p) = \frac{q}{p} G(pq, 1)$

Directed Compact Percolation near a wet wall

$$G(z,1) = \frac{1-2z-\sqrt{1-4z}}{2z}$$

$$Q(p) = \frac{q}{p}G(pq,1)$$

$$= \frac{1-2p(1-p)-|2p-1|}{2p^2}$$

$$P(p) = \begin{cases} 0, & p \leq \frac{1}{2} \\ \frac{2p-1}{p^2}, & p > \frac{1}{2} \end{cases}$$

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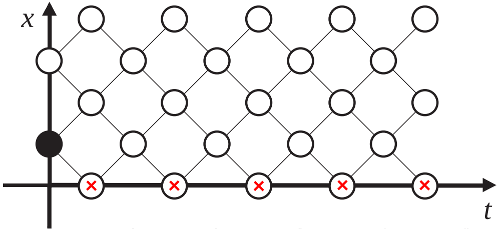

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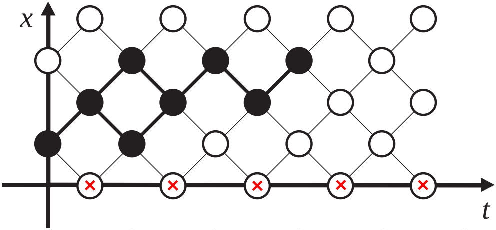

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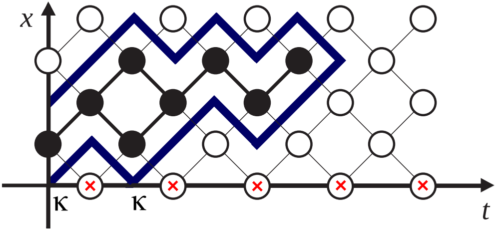
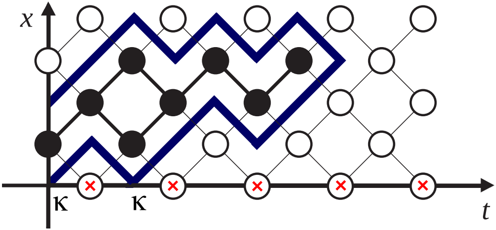


- relate to pairs of non-intersecting directed walks.
- add a weighting of κ for each contact with the wall
- $G(z, \kappa) =$ generating function for vesicles

Directed Compact Percolation near a dry wall



- relate to pairs of non-intersecting directed walks.
- add a weighting of κ for each contact with the wall
- $G(z, \kappa) =$ generating function for vesicles
- $Q(p) = \frac{1}{p^2} G(pq, \frac{1}{q})$

Directed Compact Percolation near a dry wall

$$G(z,\kappa) = \frac{z^2\kappa(\kappa-2)}{(\kappa-1)^2}\left[1+\left(1+\frac{\omega}{z}\right)\left(\frac{\omega-2z^2-\sqrt{\omega(\omega-4z^2)}}{2z^2}\right)\right]\theta(\kappa-2)$$

$$+\frac{z^2}{\kappa-1}\sum_{r=0}^{\infty} z^{2r}(C_r+zC_{r+1})\sum_{s=r+1}^{\infty} C_s\omega^{s-r}, \quad \text{where} \quad \omega=\frac{\kappa-1}{\kappa^2}$$

$$C_r = r^{th} \text{ Catalan number } = \frac{1}{r+1}\binom{2r}{r}$$

Directed Compact Percolation near a dry wall

$$G(z,\kappa) = \frac{z^2\kappa(\kappa-2)}{(\kappa-1)^2}\left[1+\left(1+\frac{\omega}{z}\right)\left(\frac{\omega-2z^2-\sqrt{\omega(\omega-4z^2)}}{2z^2}\right)\right]\theta(\kappa-2)$$

$$+\frac{z^2}{\kappa-1}\sum_{r=0}^{\infty} z^{2r}(C_r+zC_{r+1})\sum_{s=r+1}^{\infty} C_s\omega^{s-r}, \quad \text{where} \quad \omega = \frac{\kappa-1}{\kappa^2}$$

$$C_r = r^{th}\text{ Catalan number} = \frac{1}{r+1}\binom{2r}{r}$$

$$Q(p) = \frac{1}{p^2}G(pq,\frac{1}{q})$$

$$\kappa = \frac{1}{q}, \quad \omega = pq, \quad z = pq$$

Directed Compact Percolation near a dry wall

$$P(p) = \begin{cases} 0, & p \leq \frac{1}{2} \\ \frac{(2p-1)^2}{p^3}, & p > \frac{1}{2} \end{cases}$$

critical exponent $\beta = 2$

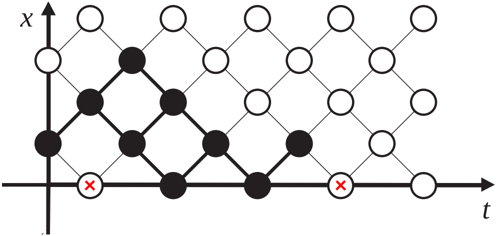
Directed Compact Percolation near a damp wall

- interpolates between wet and dry wall cases
- a wall site is wet with probability p_w, dry with probability $q_w = 1 - p_w$
- wet wall: $p_w = 1$, dry wall: $p_w = 0$.

Directed Compact Percolation near a damp wall

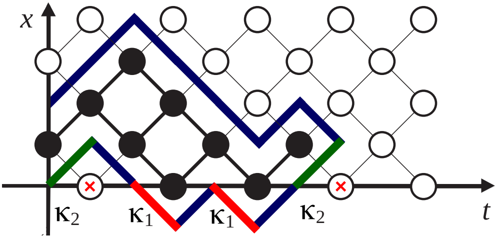
- interpolates between wet and dry wall cases
- a wall site is wet with probability p_w, dry with probability $q_w = 1 - p_w$
- wet wall: $p_w = 1$, dry wall: $p_w = 0$.
- relate to pairs of non-intersecting directed walks
  - weighting with z for each step,
  - weighting with κ_1 for wet wall sites,
  - weighting with κ_2 for dry wall sites.

Directed Compact Percolation near a damp wall



$$\text{probability} = pq_w \; p^2 \; p_w q \; q \; p_w q \; p \; qq_w = p_w^2 q_w^2 p^4 q^4$$

Directed Compact Percolation near a damp wall



probability $= pq_w \; p^2 \; p_w q \; q \; p_w q \; p \; qq_w = p_w^2 q_w^2 p^4 q^4$

weighting $= \kappa_1^2 \kappa_2^2 z^6$

Directed Compact Percolation near a damp wall

$$Q(p, p_w) \quad = \quad \text{sum of probabilities of finite clusters}$$

$$G(z, \kappa_1, \kappa_2) \quad = \quad \text{generating function for pairs of walks}$$

$$Q(p, p_w) \quad = \quad q^2 G\left(pq, \frac{p_w}{pq}, \frac{q_w}{q}\right)$$

Method of solving

$$G(z, \kappa_1, \kappa_2) = \sum_{t \geq 0} \acute{Z}_t^{\nu}(\kappa_1, \kappa_2) z^t$$

where:

$\acute{Z}_t^{\nu}(\kappa_1, \kappa_2) =$ partition function for vesicles with a free end (vesicles ending at any point, after t time steps)

Method of solving

$$\acute{Z}_t^{\nu}(\kappa_1, \kappa_2) = \sum_{x=0}^{t+1} Z_t^{\nu}(x|1)$$

where:

$Z_t^{\nu}(x|1)$ = partition function for vesicles with a fixed end (vesicles ending at x after t time steps)

Method of solving

- obtain expression for $Z_t^{\nu}(x|1)$ in terms of single walk partition function $Z_t^s(x|1)$ using Gessel-Viennot determinant:

$$Z_t^{\nu}(x|1) = \frac{1}{\kappa_2} \begin{vmatrix} Z_t^s(x|1) & Z_t^s(x+2|1) \\ Z_{t+2}^s(x|1) & Z_{t+2}^s(x+2|1) \end{vmatrix}$$

Method of solving

$$\acute{Z}_t^{\nu} = \sum_{x=0}^{t+1} Z_t^{\nu}(x|1)$$

Method of solving

$$\acute{Z}_t^{\nu} = \sum_{x=0}^{t+1} Z_t^{\nu}(x|1)$$

$$\acute{Z}_{2r}^{\nu} = \frac{1}{\kappa_2}\left(C_{r+1} Z_{2r}^{s}(1|1) + (\kappa_2 - 1) Z_{2r}^{\nu}(1|1)\right)$$

$$\acute{Z}_{2r+1}^{\nu} = \frac{1}{\kappa_2}\left(C_{r+1} Z_{2r+2}^{s}(1|1)\right)$$

Method of solving

$$\acute{Z}_t^{\nu} = \sum_{x=0}^{t+1} Z_t^{\nu}(x|1)$$

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$$\acute{Z}_{2r+1}^{\nu} = \frac{1}{\kappa_2}\left(C_{r+1} Z_{2r+2}^{s}(1|1)\right)$$

Method of solving

- obtain set of partial difference equations for $Z_t^s(x|1)$
- leads to an expression for $Z_{2r}^s(1|1)$:

Method of solving

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$$Z_{2r}^s(1|1) = \frac{\kappa_2}{c-d}\left(\theta(c-1)\frac{(c+1)(c-1)}{c}\omega_c^{-r} + \sum_{s=r+1}^{\infty} C_s \omega_c^{-r} \right.$$
$$\left. -\theta(d-1)\frac{(d+1)(d-1)}{d}\omega_d^{-r} + \sum_{s=r+1}^{\infty} C_s \omega_d^{-r}\right)$$

Method of solving

- obtain set of partial difference equations for $Z^s_t(x|1)$
- leads to an expression for $Z^s_{2r}(1|1)$:

$$Z^s_{2r}(1|1) = \frac{\kappa_2}{c-d}\left(\theta(c-1)\frac{(c+1)(c-1)}{c}\omega_c^{-r} + \sum_{s=r+1}^{\infty} C_s\omega_c^{-r}\right.$$
$$\left. -\theta(d-1)\frac{(d+1)(d-1)}{d}\omega_d^{-r} + \sum_{s=r+1}^{\infty} C_s\omega_d^{-r}\right)$$

where:

$$(1-cz^2)(1-dz^2) = 1-(\kappa_1+\kappa_2-2)z^2-(\kappa_2-1)z^4,$$

$$\omega_c = \frac{c}{(c+1)^2},\ \omega_d = \frac{d}{(d+1)^2},$$

Method of solving

$$G(z,\kappa_1,\kappa_2) = \sum_{t\geq 0} \acute{Z}^{\nu}_{t}(\kappa_1,\kappa_2) z^t$$

$$\acute{Z}^{\nu}_{2r} = \frac{1}{\kappa_2}\left(C_{r+1} Z^{s}_{2r}(1|1) + (\kappa_2 - 1) Z^{\nu}_{2r}(1|1)\right)$$

$$\acute{Z}^{\nu}_{2r+1} = \frac{1}{\kappa_2}\left(C_{r+1} Z^{s}_{2r+2}(1|1)\right)$$

Method of solving

- recurrence for $Z^s_{2r}(1|1)$:

$$Z^s_{2r}(1|1) = \frac{\kappa_2(\omega_c - \omega_d)}{c - d} C_{r+1} + (\omega_c + \omega_d) Z^s_{2r+2}(1|1) - \omega_c \omega_d Z^s_{2r+4}(1|1)$$

Method of solving

- recurrence for $Z^s_{2r}(1|1)$:

$$Z^s_{2r}(1|1) = \frac{\kappa_2(\omega_c - \omega_d)}{c-d} C_{r+1} + (\omega_c + \omega_d) Z^s_{2r+2}(1|1) - \omega_c \omega_d Z^s_{2r+4}(1|1)$$

Recall:

$$Z^\nu_t(x|1) = \frac{1}{\kappa_2} \begin{vmatrix} Z^s_t(x|1) & Z^s_t(x+2|1) \\ Z^s_{t+2}(x|1) & Z^s_{t+2}(x+2|1) \end{vmatrix}$$

Method of solving

$$Z^{\mathcal{V}}_{2r}(1|1) = \frac{\omega_c - \omega_d}{\kappa_2(c-d)} \begin{vmatrix} C_{r+1} & Z^s_{2r+2}(1|1) \\ C_{r+2} & Z^s_{2r+4}(1|1) \end{vmatrix} + \omega_c \omega_d Z^{\mathcal{V}}_{2r+2}(1|1)$$

Method of solving

$$Z_{2r}^{\nu}(1|1) = \frac{\omega_c - \omega_d}{\kappa_2(c-d)} \begin{vmatrix} C_{r+1} & Z_{2r+2}^{s}(1|1) \\ C_{r+2} & Z_{2r+4}^{s}(1|1) \end{vmatrix} + \omega_c\omega_d Z_{2r+2}^{\nu}(1|1)$$

$$\sum_{r=0}^{\infty} Z_{2r}^{\nu} z^{2r} = \frac{1}{\kappa_1^2(z^2 - \omega_c\omega_d)} \sum_{r=1}^{\infty} \left(C_r Z_{2r+2}^{s}(1|1) - C_{r+1} Z_{2r}^{s}(1|1) \right) z^{2r} - \frac{\kappa_2\omega_c\omega_d}{z^2 - \omega_c\omega_d}$$

Method of solving

$$G(z,\kappa_1,\kappa_2) = \sum_{t\geq 0} \acute{Z}_t^{\nu}(\kappa_1,\kappa_2) z^t$$

$$\acute{Z}_{2r}^{\nu} = \frac{1}{\kappa_2}\left(C_{r+1} Z_{2r}^{s}(1|1) + (\kappa_2 - 1) Z_{2r}^{\nu}(1|1)\right)$$

$$\acute{Z}_{2r+1}^{\nu} = \frac{1}{\kappa_2}\left(C_{r+1} Z_{2r+2}^{s}(1|1)\right)$$

Method of solving

$$\begin{aligned}
G(z,\kappa_1,\kappa_2) \quad = \quad & \frac{1}{c-d}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_c}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_c^s + \sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_c}\right)^r \sum_{s=r+2}^{\infty} C_s\omega_c^s\right) \\
& + \frac{cd}{\kappa_2 z(c-d)^2}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_c}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_c^s - \sum_{r=0}^{\infty} C_r\left(\frac{z^2}{\omega_c}\right)^{r-1} \sum_{s=r+2}^{\infty} C_s\omega_c^s\right) \\
& - \frac{1}{c-d}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_d^s + \frac{z}{\omega_d}\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+2}^{\infty} C_s\omega_d^s\right) \\
& - \frac{cd}{\kappa_2 z(c-d)^2}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_d^s - \frac{1}{\omega_d}\sum_{r=0}^{\infty} C_r\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+2}^{\infty} C_s\omega_d^s\right) \\
& + \theta(c-1)\frac{c^2-1}{c(c-d)}\left(2\sum_{r=0}^{\infty} C_{r+1}z^r + \frac{cd}{\kappa_2 z(c-d)}\left(\sum_{r=0}^{\infty} C_{r+1}z^r - \sum_{r=0}^{\infty} C_r z^{r-1}\right)\right) \\
& - \theta(d-1)\frac{c(d^2-1)}{\kappa_2 z(c-d)^2}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r - \frac{1}{\omega_d}\sum_{r=0}^{\infty} C_r\left(\frac{z^2}{\omega_d}\right)^r\right) \\
& - \theta(d-1)\frac{d^2-1}{d(c-d)}\left(1+\frac{z}{\omega_d}\right)\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r + \frac{cd\kappa_1}{\omega_c(c-d)(1-cd)}
\end{aligned}$$

Method of solving

$$Q(p, p_w) = q^2 G\left(pq, \frac{p_w}{pq}, \frac{q_w}{q}\right)$$

ie: $z = pq, \quad \kappa_1 = \dfrac{p_w}{pq}, \quad \kappa_2 = \dfrac{q_w}{q}$

$$c = \frac{p}{q}, \quad d = \frac{p_w - p}{p}, \quad \omega_c = pq, \quad \omega_d = \frac{p}{p_w^2}(p_w - p)$$

Method of solving

$$Q(p, p_w) \;=\; q^2 G\left(pq, \frac{p_w}{pq}, \frac{q_w}{q}\right)$$

$$\text{ie:}\quad z \;=\; pq, \quad \kappa_1 \;=\; \frac{p_w}{pq}, \quad \kappa_2 \;=\; \frac{q_w}{q}$$

$$c \;=\; \frac{p}{q}, \quad d \;=\; \frac{p_w - p}{p}, \quad \omega_c \;=\; pq, \quad \omega_d \;=\; \frac{p}{p_w^2}(p_w - p)$$

$$p > \frac{1}{2}, \quad 0 \le p_w \le 1, \quad \text{so} \quad \theta(d-1) \;=\; 0$$

$$\omega_c \;=\; z$$

Introduction
Damp wall
Percolation problem
Method of solving
Results
Further work

Method of solving

$$
\begin{aligned}
G(z,\kappa_1,\kappa_2) \;=\; & \frac{1}{c-d}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_c}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_c^s + \sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_c}\right)^r \sum_{s=r+2}^{\infty} C_s\omega_c^s\right)\\
& + \frac{cd}{\kappa_2 z(c-d)^2}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_c}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_c^s - \sum_{r=0}^{\infty} C_r\left(\frac{z^2}{\omega_c}\right)^{r-1} \sum_{s=r+2}^{\infty} C_s\omega_c^s\right)\\
& - \frac{1}{c-d}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_d^s + \frac{z}{\omega_d}\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+2}^{\infty} C_s\omega_d^s\right)\\
& - \frac{cd}{\kappa_2 z(c-d)^2}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_d^s - \frac{1}{\omega_d}\sum_{r=0}^{\infty} C_r\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+2}^{\infty} C_s\omega_d^s\right)\\
& + \theta(c-1)\frac{c^2-1}{c(c-d)}\left(2\sum_{r=0}^{\infty} C_{r+1}z^r + \frac{cd}{\kappa_2 z(c-d)}\left(\sum_{r=0}^{\infty} C_{r+1}z^r - \sum_{r=0}^{\infty} C_r z^{r-1}\right)\right)\\
& - \theta(d-1)\frac{c(d^2-1)}{\kappa_2 z(c-d)^2}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r - \frac{1}{\omega_d}\sum_{r=0}^{\infty} C_r\left(\frac{z^2}{\omega_d}\right)^r\right)\\
& - \theta(d-1)\frac{d^2-1}{d(c-d)}\left(1+\frac{z}{\omega_d}\right)\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r + \frac{cd\kappa_1}{\omega_c(c-d)(1-cd)}
\end{aligned}
$$

Method of solving

$$\begin{aligned}
G(z,\kappa_1,\kappa_2) \quad = \quad & \frac{1}{c-d}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_c}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_c^s + \sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_c}\right)^r \sum_{s=r+2}^{\infty} C_s\omega_c^s\right) \\
& + \frac{cd}{\kappa_2 z(c-d)^2}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_c}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_c^s - \sum_{r=0}^{\infty} C_r\left(\frac{z^2}{\omega_c}\right)^{r-1} \sum_{s=r+2}^{\infty} C_s\omega_c^s\right) \\
& - \frac{1}{c-d}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_d^s + \frac{z}{\omega_d}\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+2}^{\infty} C_s\omega_d^s\right) \\
& - \frac{cd}{\kappa_2 z(c-d)^2}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_d^s - \frac{1}{\omega_d}\sum_{r=0}^{\infty} C_r\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+2}^{\infty} C_s\omega_d^s\right) \\
& + \theta(c-1)\frac{c^2-1}{c(c-d)}\left(2\sum_{r=0}^{\infty} C_{r+1}z^r + \frac{cd}{\kappa_2 z(c-d)}\left(\sum_{r=0}^{\infty} C_{r+1}z^r - \sum_{r=0}^{\infty} C_r z^{r-1}\right)\right) \\
& + \frac{cd\kappa_1}{\omega_c(c-d)(1-cd)}
\end{aligned}$$

Method of solving

$$
\begin{aligned}
G(z,\kappa_1,\kappa_2) \quad = \quad & \frac{1}{c-d}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_c}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_c^s + \sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_c}\right)^r \sum_{s=r+2}^{\infty} C_s\omega_c^s\right) \\
& + \frac{cd}{\kappa_2 z(c-d)^2}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_c}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_c^s - \sum_{r=0}^{\infty} C_r\left(\frac{z^2}{\omega_c}\right)^{r-1} \sum_{s=r+2}^{\infty} C_s\omega_c^s\right) \\
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& + \theta(c-1)\frac{c^2-1}{c(c-d)}\left(2\sum_{r=0}^{\infty} C_{r+1}z^r + \frac{cd}{\kappa_2 z(c-d)}\left(\sum_{r=0}^{\infty} C_{r+1}z^r - \sum_{r=0}^{\infty} C_r z^{r-1}\right)\right)
\end{aligned}
$$

$$
+\frac{cd\kappa_1}{\omega_c(c-d)(1-cd)}
$$

Method of solving

$$
\begin{aligned}
G(z,\kappa_1,\kappa_2) \;=\;& \frac{1}{c-d}\left(\sum_{r=0}^{\infty} C_{r+1}z^r \sum_{s=r+1}^{\infty} C_s\omega_c^s + \sum_{r=0}^{\infty} C_{r+1}z^r \sum_{s=r+2}^{\infty} C_s\omega_c^s\right)\\
&+\frac{cd}{\kappa_2 z(c-d)^2}\left(\sum_{r=0}^{\infty} C_{r+1}z^r \sum_{s=r+1}^{\infty} C_s\omega_c^s - \sum_{r=0}^{\infty} C_r z^{r-1} \sum_{s=r+2}^{\infty} C_s\omega_c^s\right)\\
&-\frac{1}{c-d}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_d^s + \frac{z}{\omega_d}\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+2}^{\infty} C_s\omega_d^s\right)\\
&-\frac{cd}{\kappa_2 z(c-d)^2}\left(\sum_{r=0}^{\infty} C_{r+1}\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+1}^{\infty} C_s\omega_d^s - \frac{1}{\omega_d}\sum_{r=0}^{\infty} C_r\left(\frac{z^2}{\omega_d}\right)^r \sum_{s=r+2}^{\infty} C_s\omega_d^s\right)\\
&+\theta(c-1)\frac{c^2-1}{c(c-d)}\left(2\sum_{r=0}^{\infty} C_{r+1}z^r + \frac{cd}{\kappa_2 z(c-d)}\left(\sum_{r=0}^{\infty} C_{r+1}z^r - \sum_{r=0}^{\infty} C_r z^{r-1}\right)\right)
\end{aligned}
$$

$$
+\frac{cd\kappa_1}{\omega_c(c-d)(1-cd)}
$$

Percolation probability for damp wall

$$P(p, p_w) = \frac{(2p-1)^2}{p^2(p - p_w + pp_w)}, \quad p > \frac{1}{2}$$

Percolation probability for damp wall

$$P(p, p_w) = \begin{cases} 0 & p \leq \frac{1}{2} \\ \frac{(2p-1)^2}{p^2(p-p_w+pp_w)} & p > \frac{1}{2} \end{cases}$$

$$\text{wet:} \quad P(p,1) = \frac{2p-1}{p^2}, \quad p > \frac{1}{2}$$

$$\text{dry:} \quad P(p,0) = \frac{(2p-1)^2}{p^3}, \quad p > \frac{1}{2}$$

- same critical exponent as in dry wall case, $\beta = 2$ (except for $p_w = 1$, the wet wall case, with $\beta = 1$)

Mean length of finite clusters

Bulk case:

$$L(p) = q^2 \frac{d}{dz} \left(zG(z) \right) \Big|_{z=pq}$$

$$L(p) = \frac{1}{|2p-1|}$$

Mean length of finite clusters

Wet wall case:

$$L(p) = q^2 \frac{\partial}{\partial z} \left(z G(z, \kappa) \right) \Big|_{z=pq,\ \kappa=1}$$

$$L(p) = \frac{1}{|2p-1|}$$

Mean length of finite clusters

Dry wall case:

$$
\begin{aligned}
L(p) &= \frac{q^2}{p}\sum_{r=0}^{\infty}(2r+1)C_r z^r \sum_{s=r+1}^{\infty} C_s z^s \\
&\quad + \frac{q^2}{p}\sum_{r=0}^{\infty}(2r+2)C_{r+1} z^{r+1} \sum_{s=r+1}^{\infty} C_s z^s \\
&\quad + \theta(p-p_c)\frac{q(3-2p)}{p^3}
\end{aligned}
$$

Mean length of finite clusters

Dry wall case:

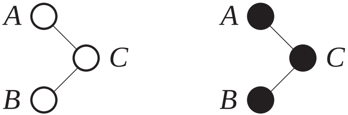
$$L(p) = \frac{1}{8p^3}\left(-5 + 4z + 6\sqrt{1-4z} - \frac{8E(16z^2)}{\pi} + \frac{2(3-4z)(1+4z)K(16z^2)}{\pi}\right) + \theta(p-p_c)\frac{q(3-2p)}{p^3}$$

where $z = p(1-p)$

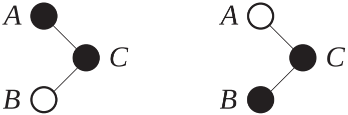
Further work

- mean length of finite clusters
- mean number of wall contacts for finite clusters
- solving problem for general seed width m (currently using $m = 1$)
- investigating effect of bias towards or away from the wall – introducing p_u and p_d

Further work

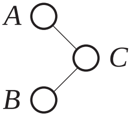


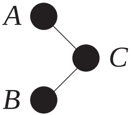
Pr(C occupied): 0 1



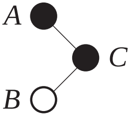
Pr(C occupied): p p

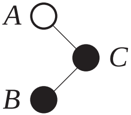
Further work





Pr(C occupied): 0 1





Pr(C occupied): p_d p_u

Further work

- mean length of finite clusters
- mean number of wall contacts for finite clusters
- solving problem for general seed width m (currently using $m = 1$)
- investigating effect of bias towards or away from the wall – introducing p_u and p_d