Journal of Physics A: Mathematical and Theoretical

PAPER • **OPEN ACCESS**

Exact solution of weighted partially directed walks crossing a square

To cite this article: Nicholas R Beaton and Aleksander L Owczarek 2023 *J. Phys. A: Math. Theor.* **56** 155003

OPEN ACCESS
IOP Publishing

Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **56** (2023) 155003 (20pp)

https://doi.org/10.1088/1751-8121/acc2fd

Exact solution of weighted partially directed walks crossing a square

Nicholas R Beaton* and **Aleksander L Owczarek**

School of Mathematics and Statistics, The University of Melbourne, Parkville, VIC 3010, Australia

E-mail: nrbeaton@unimelb.edu.au

Received 18 December 2022; revised 6 March 2023
Accepted for publication 9 March 2023
Published 21 March 2023

CrossMark

Abstract

We consider partially directed walks crossing a $L \times L$ square weighted according to their length by a fugacity t. The exact solution of this model is computed in three different ways, depending on whether t is less than, equal to or greater than 1. In all cases a complete expression for the dominant asymptotic behaviour of the partition function is calculated. The model admits a dilute to dense phase transition, where for $0 < t < 1$ the partition function scales exponentially in L whereas for $t > 1$ the partition function scales exponentially in L^2, and when $t = 1$ there is an intermediate scaling which is exponential in $L \log L$. As such we provide an exact solution of a model of the dilute to dense polymeric phase transition in two dimensions.

Keywords: self-avoiding walk, exact solutions, polymers, phase transition

(Some figures may appear in colour only in the online journal)

1. Introduction

The problem of self-avoiding walks (SAWs) crossing a square [2, 13, 18, 27], or walks or polygons simply contained in a square [3, 9, 10] in two dimensions, or inside a cubic box in three dimensions [26], has attracted attention over an extended period including recently, with various rigorous and numerical (Monte Carlo and series analysis) results being accumulated. These problems provide a simple model of a confined polymer which illustrate a different lens through which to consider single polymer behaviour. When a length fugacity is added to the

* Author to whom any correspondence should be addressed.

basic set-up the models can be shown to demonstrate a phase transition between a dilute phase for low fugacity and a dense phase for large fugacity [2, 18, 27]. The scaling of the partition function is fundamentally different in these two regimes, with exponential scaling linear in the side of the square (box) in the dilute phase and exponential in the area of the square (volume of the box) in the dense phase.

Let $c_{L,n}$ be the number of n-step SAWs on the square lattice which cross an $L \times L$ square from the south-west corner to the north-east corner, and define the partition function

$$C_L(t) = \sum_n c_{L,n} t^n. \tag{1.1}$$

Then it is known rigorously (e.g. [18, 27]) that the limits

$$\lambda_1(t) = \lim_{L\to\infty} C_L(t)^{1/L} \tag{1.2}$$

$$\lambda_2(t) = \lim_{L\to\infty} C_L(t)^{1/L^2} \tag{1.3}$$

exist or are infinite. More precisely, $\lambda_1(t)$ is finite for $0 < t \leqslant \mu^{-1}$ and infinite for $t > \mu^{-1}$, where μ is the connective constant of the lattice; and $\lambda_2(t) = 1$ for $0 < t \leqslant \mu^{-1}$ and is finite and > 1 for $t > \mu^{-1}$. Moreover $\lambda_1(t) < 1$ for $t < \mu^{-1}$ and $\lambda_1(\mu^{-1}) = 1$; otherwise the values of $\lambda_1(t)$ and $\lambda_2(t)$ are not known for $t < \mu^{-1}$ and $t > \mu^{-1}$ respectively. These results generalise to higher dimensions. The precise nature of the 'subexponential' behaviour of $C_L(t)$ is not known, however it has been recently shown by Whittington [26] that

$$C_L(1) = \lambda^{L^2 + O(L)} \tag{1.4}$$

with $\lambda = \lambda_2(1)$. A similar result holds for higher dimensions. This accords with the conjecture [9, 10] that

$$C_L(1) \sim \lambda^{L^2 + bL + c} L^g \tag{1.5}$$

for constants b, c and g. Note that here and below in the sequel the notation $a_L \sim b_L$ indicates that $\lim_{L\to\infty} \frac{a_L}{b_L} = 1$.

Burkhardt and Guim [5] used the connection between SAWs and the $N \to 0$ limit of the $O(N)$ model to conjecture the finite-size scaling form

$$C_L(t) \approx L^{-\eta_c} f[L^{1/\nu}(t - \mu^{-1})], \qquad t \approx t_c \tag{1.6}$$

where f is a scaling function, $\nu = \frac{3}{4}$ is the metric exponent for SAWs (in two dimensions) and $\eta_c = \frac{5}{2}$ is the corner exponent (for a 90° corner) of the magnetisation of the $O(N)$ model, as per Cardy [6]. Consequently it is expected that

$$C_L(t_c) \sim \text{const} \cdot L^{-\eta_c}, \qquad L \to \infty. \tag{1.7}$$

Numerical evidence [2] agrees with this conjecture.

One can instead first take the thermodynamic limit and then consider the scaling of the resulting limiting free energy. With $f_i(t) = \log \lambda_i(t)$ it is known [18] that

$$f_1(t) \sim m(t), \quad t \to (\mu^{-1})^- \tag{1.8}$$

where $m(t)$ is the so-called *mass* of SAWs (see e.g. [18, 19] for a precise definition). This function is known to exist but its precise form is unknown; it is however conjectured to behave as

$$m(t) \sim \text{const} \cdot (\mu^{-1} - t)^{\nu}, \qquad t \to (\mu^{-1})^-. \tag{1.9}$$

Similarly when t approaches the critical point from above it is believed (see again [18]) that

$$f_2(t) \sim \text{const} \cdot (t - \mu^{-1})^{\mathrm{d}\nu}, \qquad t \to (\mu^{-1})^+. \tag{1.10}$$

It is natural to also consider the average length $\langle n \rangle_L(t)$ of SAWs in the $L \times L$ box, according to the Boltzmann distribution which assigns probability $t^{|\omega|}/C_L(t)$ to each walk ω. This is given by

$$\langle n \rangle_L(t) = \frac{t\frac{\mathrm{d}}{\mathrm{d}t} C_L(t)}{C_L(t)}. \tag{1.11}$$

Then it is known [27] that as $L \to \infty$

$$\langle n \rangle_L(t) = \begin{cases} \Theta(L), & t < \mu^{-1} \\ \Theta(L^2), & t > \mu^{-1} \end{cases} \tag{1.12}$$

where $f(x) = \Theta(g(x))$ means that there exist constants c_1, c_2 such that $c_1 g(x) \leqslant f(x) \leqslant c_2 g(x)$. At the critical point, the conjectured scaling form (1.6) implies that

$$\langle n \rangle_L(\mu^{-1}) \sim \text{const} \cdot L^{1/\nu}. \tag{1.13}$$

Here we consider a variation of this model, namely partially directed walks (PDWs) crossing an $L \times L$ square. These are walks which take steps $(1,0), (0,1)$ and $(0,-1)$ while remaining self-avoiding. This is, of course, a simpler model than SAWs, but directed and partially directed walks have been shown to display complex critical behaviour for a range of models, from adsorption to collapse (see e.g. [4, 8, 11, 12, 15, 17, 20, 21, 23, 24, 28]). Here we compute the exact solution of PDWs crossing a square and provide the *full dominant asymptotics* of the partition function and average number of steps as a function of the length fugacity t.

For PDWs the dilute-dense phase transition occurs at $t = 1$. Interestingly, each regime (dilute, dense, and at the critical point) requires a different mathematical approach to elucidate the solution. For small $t < 1$ the generating function is found via the kernel method, and the asymptotics of the partition function follow via saddle point methods. For large $t > 1$ a transfer matrix method is required, and is analysed with a Bethe ansatz type solution and the asymptotics follow a subtle analysis of the Bethe roots. The solution at $t = 1$ is simply found via a direct combinatorial argument.

The structure of the remainder of the paper is as follows. In section 2 we define the model of interest and state the main results, namely theorem 1 (the asymptotics of the partition function) and lemma 1 (the asymptotics of the mean number of steps). Section 3 covers the unweighted case $t = 1$. In section 4 we focus on the dilute case $t < 1$ and in section 5 the dense case $t > 1$. Finally section 6 contains some further discussion.

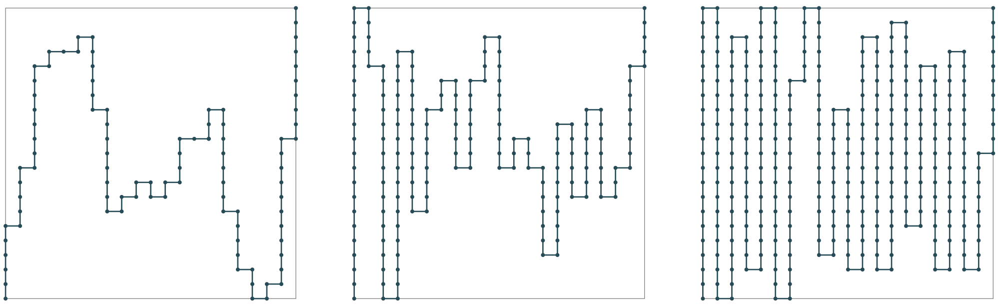
2. Model and central results

Let $\mathcal{P}_{L,n}$ be the set of n-step PDWs which cross an $L \times L$ square from the south-west corner to the north-east corner, and let $p_{L,n} = |\mathcal{P}_{L,n}|$. Define the partition function

$$P_L(t) = \sum_n p_{L,n} t^n. \tag{2.1}$$

For a given value of $t > 0$, the Boltzmann distribution on $\mathcal{P}_L = \bigcup_n \mathcal{P}_{L,n}$ assigns probability

$$\mathbb{P}_L(t, \omega) = \frac{t^{|\omega|}}{P_L(t)} \tag{2.2}$$



(a) (b) (c)

Figure 1. PDWs in a box of size $L = 20$ sampled from the Boltzmann distribution, at (a) $t = 0.8$, (b) $t = 1$ and (c) $t = 1.2$. The respective lengths are 92, 170 and 326.

to the PDW ω, where $|\omega|$ is the length of ω. See figure 1 for some PDWs in the box of size $L = 20$ sampled from the Boltzmann distribution at various values of t.

We then define the mean number of steps for walks in the $L \times L$ square to be

$$\langle n \rangle_L(t) = \frac{\sum_n n p_{L,n} t^n}{\sum_n p_{L,n} t^n} = \frac{t \frac{\mathrm{d}}{\mathrm{d}t} P_L(t)}{P_L(t)}. \tag{2.3}$$

Our main result is the following.

Theorem 1. *The partition functions $P_L(t)$ satisfy the following.*

(i) For $t = 1$,

$$P_L(1) = (L+1)^L \sim e \cdot e^{L \log L}. \tag{2.4}$$

(ii) For $0 < t < 1$,

$$P_L(t) \sim \frac{1}{\sqrt{\pi}} \cdot \left(\frac{1-t^2}{1+t^2} \right)^2 \cdot L^{-1/2} \cdot \left(\frac{4t^2}{1-t^2} \right)^L. \tag{2.5}$$

(iii) For $t > 1$,

$$P_L(t) \sim \begin{cases} \left(\dfrac{t^4}{t^2-1} \right)^L t^{L^2} & L \text{ even} \\ \dfrac{t^2-1}{t^2} \cdot L^2 \cdot \left(\dfrac{t^3}{t^2-1} \right)^L \cdot t^{L^2} & L \text{ odd.} \end{cases} \tag{2.6}$$

See figure 2 for plots of $P_L(t)$ for $t = \frac{1}{2}$ and $t = 2$.

Lemma 1. *The mean number of steps $\langle n \rangle_L(t)$ satisfies the following.*

(i) For $t = 1$,

$$\langle n \rangle_L(1) = \frac{L(L^2 + 7L + 4)}{3(L+1)} \sim \frac{L^2}{3} + 2L - \frac{2}{3}. \tag{2.7}$$

(ii) For $0 < t < 1$,

$$\langle n \rangle_L(t) \sim \frac{2L}{1-t^2} - \frac{8t^2}{1-t^4}. \tag{2.8}$$

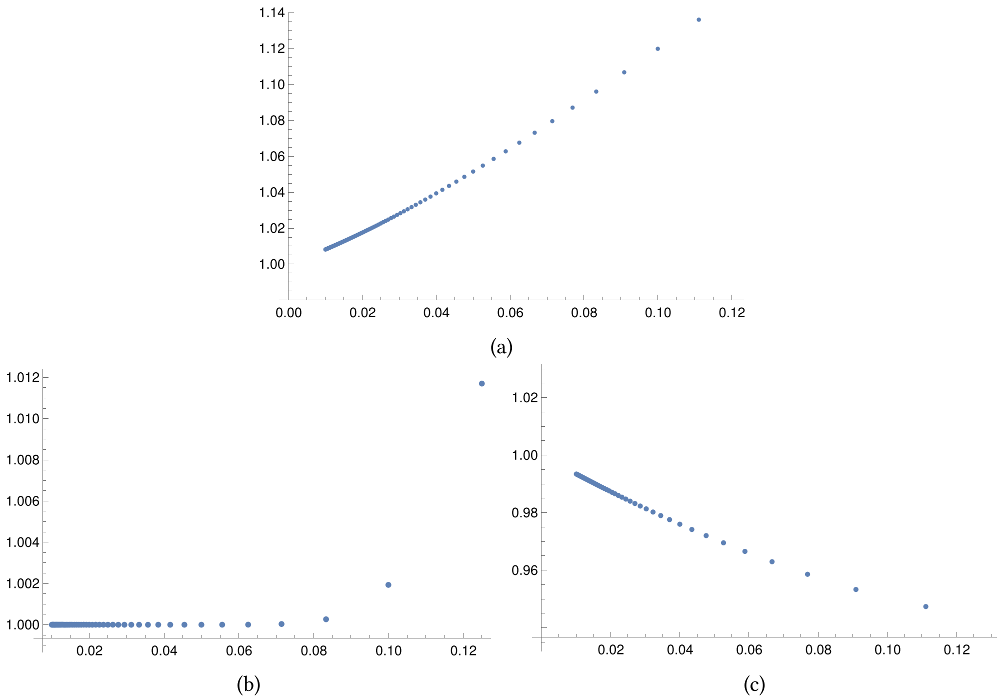


Figure 2. (a) Plot of $P_L(t)$ divided by the expression (2.5) against $\frac{1}{L}$ at $t = \frac{1}{2}$ for L up to 100. (b)–(c) Plots of $P_L(t)$ divided by the expression (2.6) against $\frac{1}{L}$ at $t = 2$ for L up to 100, for (b) even L and (c) odd L.

(iii) For $t > 1$,

$$\langle n \rangle_L(t) \sim \begin{cases} L^2 + \dfrac{2(t^2-2)L}{t^2-1} & L \text{ even} \\ L^2 + \dfrac{(t^2-3)L}{t^2-1} + \dfrac{2}{t^2-1} & L \text{ odd.} \end{cases} \tag{2.9}$$

Parts (ii) and (iii) of lemma 1 follow by applying (2.3) to the respective results in theorem 1. Part (i) follows by applying (2.3) to (5.2). See figure 3 for plots of $\langle n \rangle_L(t)$ for $t = \frac{1}{2}$, $t = 1$ and $t = 2$.

3. The unweighted case $t = 1$

When $t = 1$ we are simply interested in counting the number of PDWs in the box. Let $\mathcal{P}_L$ be the set of PDWs which cross the $L \times L$ box from the bottom left corner to the top right corner. Then there is a simple bijection between $\mathcal{P}_L$ and the set

$$\mathcal{W}_L := \{(w_1, w_2, \ldots, w_L) \in \mathbb{Z}^L : 0 \leqslant w_i \leqslant L\}, \tag{3.1}$$

where we encode a PDW by the heights of its horizontal steps, reading left to right.

Clearly

$$|\mathcal{W}_L| = P_L(1) = (L+1)^L. \tag{3.2}$$

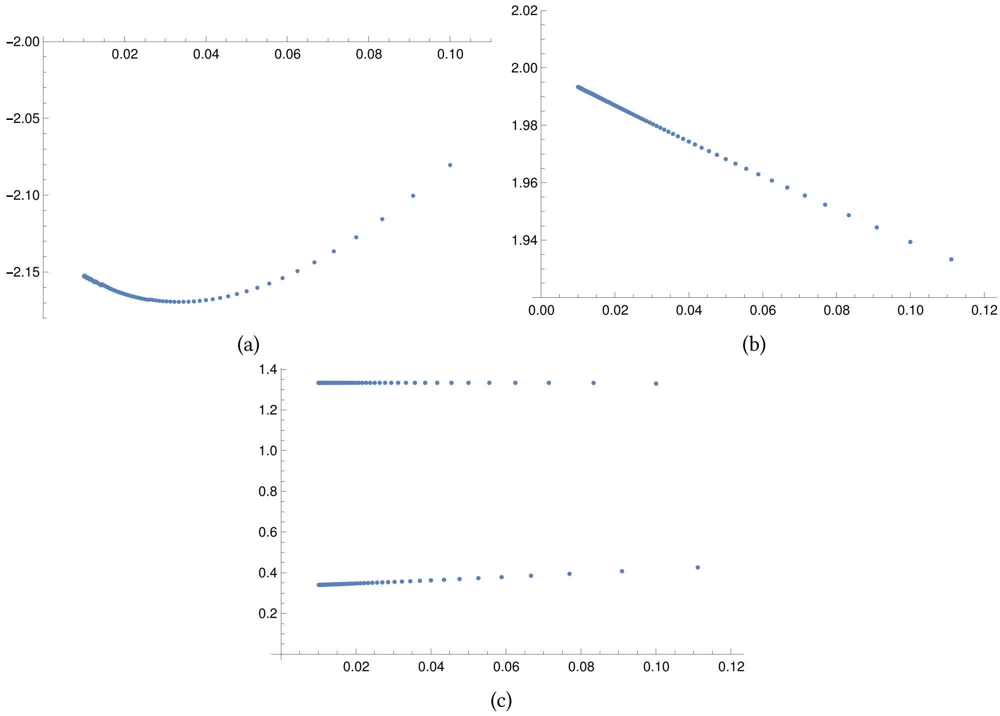


Figure 3. (a) Plot of $\langle n\rangle_L - \frac{8L}{3}$ against $\frac{1}{L}$ at $t = \frac{1}{2}$ for L up to 100. The points are approaching $-\frac{8t^2}{1-t^4} = -\frac{32}{15}$. (b) Plot of $\frac{1}{L}(\langle n\rangle_L - \frac{L^2}{3})$ against $\frac{1}{L}$ at $t = 1$ for L up to 100. (c) Plot of $\frac{1}{L}(\langle n\rangle_L - L^2)$ against $\frac{1}{L}$ at $t = 2$ for L up to 100. For even L the points are approaching $\frac{2(t^2-2)}{t^2-1} = \frac{4}{3}$ and for odd L they are approaching $\frac{t^2-3}{t^2-1} = \frac{1}{3}$.

We thus have neither λ_1^L nor $\lambda_2^{L^2}$ growth, but instead something in between, namely

$$P_L(1) = e\cdot e^{L\log L}\left(1 - \frac{1}{2L} + \frac{11}{24L^2} + O(L^{-3})\right) \tag{3.3}$$

which establishes theorem 1 (i).

Note that this method is of no use when computing $\langle n\rangle_L$ at $t = 1$. To do this we use the expression (5.2), taking its derivative and setting $t = 1$.

4. The dilute case $t < 1$

4.1. Computing generating functions

For the dilute case we will compute the generating function using the kernel method and derive the asymptotics using the saddle point method. The origin of the kernel method is often attributed to Knuth [14]; the particular version we use here is similar to the implementation in [22].

We generalise from PDWs crossing a box to PDWs in a strip, i.e. $S_L = \{(x,y)\in\mathbb{Z}^2 : 0 \leqslant y \leqslant L\}$. The walks all start at $(0,0)$. We use three generating functions:

- $H(t,s,v) \equiv H(v)$: Counts the empty walk and walks ending with a horizontal step, with t conjugate to length, s conjugate to horizontal span (i.e. number of horizontal steps) and v conjugate to the height of the endpoint.
- $U(t,s,v) \equiv U(v)$: Counts walks ending with an up step.
- $D(t,s,v) \equiv D(v)$: Counts walks ending with a down step.

Then by appending one step at a time, we have the functional equations

$$H(v) = 1 + ts\left(H(v) + U(v) + D(v)\right), \tag{4.1}$$

$$U(v) = tv\left(H(v) + U(v)\right) - tv\left(v^L[v^L]H(v) + v^L[v^L]U(v)\right), \tag{4.2}$$

$$D(v) = t\bar{v}\left(H(v) + D(v)\right) - t\bar{v}\left([v^0]H(v) + [v^0]D(v)\right), \tag{4.3}$$

where the notation $[v^h]G(v)$ means to take the coefficients of all terms which contain v^h from the generating function $G(v)$. Additionally by considering the bottom and top boundaries, we have

$$[v^0]H(v) = 1 + ts\left([v^0]H(v) + [v^0]D(v)\right) \tag{4.4}$$

$$H(v) = ts\left([v^L]H(v) + [v^L]U(v)\right). \tag{4.5}$$

Combining all the above and eliminating all the U and D terms gives

$$\left(1 - ts + \frac{t^2 s}{t-v} - \frac{t^2 sv}{1-tv}\right)H(v) = 1 - \frac{t}{t-v} + \frac{t}{t-v}H_0 - \frac{tv^{L+1}}{1-tv}H_L \tag{4.6}$$

where $H_0 = [v^0]H(v)$ and $H_L = [v^L]H(v)$.

We now apply the *kernel method* to solve this equation. The kernel is

$$K(t,s,v) \equiv K(v) = 1 - ts + \frac{t^2 s}{t-v} - \frac{t^2 sv}{1-tv} \tag{4.7}$$

which has two roots in v, namely

$$v = V = \frac{1 - ts + t^2 + t^3 s - \sqrt{(1 - ts + t^2 + t^3 s)^2 - 4t^2}}{2t} \tag{4.8}$$

$$= t + st^2 + s^2 t^3 + s^3 t^4 + (s^2 + s^4)t^5 + \cdots \tag{4.9}$$

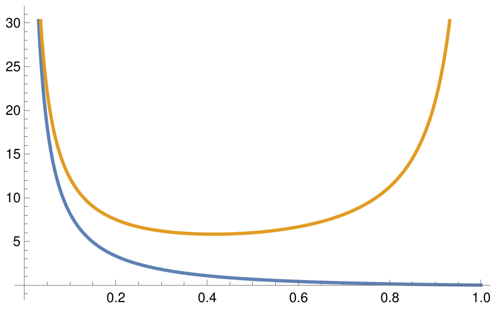
and

$$v = V^{-1} = t^{-1} - s - s^2 t^3 - s^3 t^4 - (s^2 + s^4)t^5 + \cdots. \tag{4.10}$$

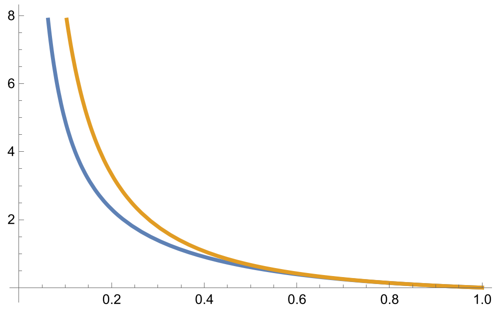
Since $H(v)$ has only finite powers of v (namely, v^0 to v^L), both of the kernel roots can be substituted into (4.6) with $H(v)$ still being a well-defined (Laurent) series in t. We thus cancel the LHS and get a pair of equations with unknowns H_0 and H_L, which can be solved. We get

$$H_L = \frac{V^L(t-V)(1-tV)(1-V^2)}{t((t-V)^2 V^{2L+2} - (1-tV)^2)} \tag{4.11}$$

and similar for H_0.



(a)



(b)

Figure 4. (a) Plot of s_1 (blue) and s_2 (orange). (b) Plot of s_0 (blue) and s_1 (orange).

4.2. Extracting coefficients

We know that for any fixed L, H_L is a rational function, though the exact way in which all the square roots cancel from (4.11) is far from obvious. To get PDWs crossing a box, we want

$$P_L(t) = t^{-1}[s^{L+1}]H_L. \tag{4.12}$$

Since H_L is rational, it is meromorphic in the complex s plane for any real (or complex) t. So we have

$$P_L(t) = \frac{1}{2\pi it}\oint \frac{H_L}{s^{L+2}}\mathrm{d}s \tag{4.13}$$

where the contour integral is a simple closed curve around the origin.

The form of (4.11) is not particularly conducive to computing the above contour integral. Let us rewrite it slightly as

$$H_L = \frac{V^L(t-V)(1-V^2)}{-t(1-tV)}\cdot\frac{1}{1-(t-V)^2V^{2L+2}/(1-tV)^2}. \tag{4.14}$$

In taking the contour integral we may assume that $|s|$ is small (the exact radius will be determined shortly) so that $|V|$ is close to t. Then

$$\left|\frac{(t-V)^2V^{2L+2}}{(1-tV)^2}\right| \sim |s^2|t^{2L+6} \tag{4.15}$$

for large L. This is thus small, and so we can approximate H_L as

$$H_L \sim H_L^* = \frac{V^L(t-V)(1-V^2)}{-t(1-tV)}. \tag{4.16}$$

However, we now have a problem. H_L was a rational (i.e. meromorphic) function but H_L^* is not. So there may now be branch cuts to contend with. These arise from the square root term in V, which is

$$\sqrt{(1-ts+t^2+t^3s)^2-4t^2} \tag{4.17}$$

The term inside the square root is 0 at

$$s_1 = \frac{1-t}{t(1+t)} \qquad \text{and} \qquad s_2 = \frac{1+t}{t(1-t)}. \tag{4.18}$$

We have $0 < s_1 < s_2$ for $t \in (0,1)$, with $s_1 \to 0$ as $t \to 1$. See figure 4(a).

The term inside the square root is negative for $s_1 < s < s_2$ and positive (for real s) for $s < s_1$ and $s > s_2$. We may thus place the branch cut along the real axis between s_1 and s_2, and as long as our contour integral is along a curve with $|s| < s_1$ then we avoid the branch cut.

Next we need to check if H_L^* has any poles that we need to take into consideration. From the form of V we can see that the numerator presents no problem. For the denominator we need to check only $(1-tV)$, but a bit of rearranging shows that this has no roots in s.

So it remains to compute the asymptotics of

$$P_L^*(t) = \frac{1}{2\pi i t}\oint \frac{H_L^*}{s^{L+2}}\mathrm{d}s = \frac{1}{2\pi i t}\oint \frac{V^L(t-V)(1-V^2)}{-t(1-tV)s^{L+2}}\mathrm{d}s \tag{4.19}$$

where the contour has to be within $|s| < s_1$.

4.3. Asymptotics via the saddle point method

The most basic form of the saddle point method gives

$$\int g(z)\exp(nh(z))\mathrm{d}z \sim i\sqrt{\frac{2\pi}{nh''(z_0)}}g(z_0)\exp(nh(z_0)), \qquad n\to\infty \tag{4.20}$$

where z_0 is a saddle point of $h(z)$.

The form (4.19) is well set up for estimation using the saddle point method. The dependence on L is from

$$\left(\frac{V}{s}\right)^L = \exp(Lh(s)) \tag{4.21}$$

where $h(s) = \log V - \log s$. h has a saddle point at

$$s_0 = \frac{1-t^2}{2t(1+t^2)}. \tag{4.22}$$

It is straightforward to check that $0 < s_0 < s_1$ for $0 < t < 1$ (see figure 4(b)). Both $s_0, s_1 \to 0$ as $t\to 1$.

For us

$$g(s) = \frac{(t-V)(1-V^2)}{-t(1-tV)s^2} \tag{4.23}$$

Substituting,

$$g(s_0) = 4t^2. \tag{4.24}$$

Meanwhile

$$\exp(h(s_0)) = \frac{4t^2}{1-t^2} \tag{4.25}$$

$$h''(s_0) = \frac{8t^2(1+t^2)^4}{(1-t^2)^4}. \tag{4.26}$$

Putting this all together,

$$P_L^*(t) \sim \frac{1}{2\pi i t}\cdot i 4t^2 L^{-1/2}\sqrt{2\pi\cdot\frac{(1-t^2)^4}{8t^2(1+t^2)^4}}\left(\frac{4t^2}{1-t^2}\right)^L \tag{4.27}$$

$$= \frac{1}{\sqrt{\pi}}\cdot\left(\frac{1-t^2}{1+t^2}\right)^2\cdot L^{-1/2}\cdot\left(\frac{4t^2}{1-t^2}\right)^L \tag{4.28}$$

as in theorem 1 (ii).

5. The dense case $t > 1$

5.1. Transfer matrix formulation and Bethe ansatz solution

For the dense case we must use a completely different method to compute asymptotics, using a transfer matrix approach. Define the $(L+1)\times(L+1)$ matrix

$$T_L(t) = \begin{pmatrix} t & t^2 & t^3 & \cdots & t^{L+1} \\ t^2 & t & t^2 & \cdots & t^L \\ t^3 & t^2 & t & \cdots & t^{L-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t^{L+1} & t^L & t^{L-1} & \cdots & t \end{pmatrix}. \tag{5.1}$$

Then

$$P_L(t) = (1,t,t^2,\ldots,t^L)\cdot T_L(t)^L \cdot (0,0,\ldots,0,1)^{\mathsf{T}} \tag{5.2}$$

$$= \frac{1}{t}(1,0,0,\ldots,0)\cdot T_L(t)^{L+1}\cdot(0,0,\ldots,0,1)^{\mathsf{T}}. \tag{5.3}$$

For brevity, in the following we may drop subscripts or functional arguments. Let us consider the eigen-equation

$$T\mathbf{g} = \lambda \mathbf{g}, \tag{5.4}$$

that is

$$\sum_{j=1}^{L+1} T_{i,j} g_j = \lambda g_i \qquad \text{for } i = 1,\ldots,L+1. \tag{5.5}$$

We begin with the ansatz $g_j = z^j$ for some complex number z, giving

$$\sum_{j=1}^{L+1} t^{|i-j|+1} z^j = \lambda z^i. \tag{5.6}$$

Splitting the sum gives

$$\sum_{j=1}^{i} t^{(i-j)+1} z^j + \sum_{j=i+1}^{L+1} t^{(j-i)+1} z^j = \lambda z^i \tag{5.7}$$

or rather

$$zt^i \sum_{k=0}^{i-1} \left(\frac{z}{t}\right)^k + t^{1-i}(tz)^{i+1} \sum_{k=0}^{L-i} (tz)^k = \lambda z^i. \tag{5.8}$$

Summing the partial geometric series, we find

$$zt^i \left[\frac{1-(\frac{z}{t})^i}{1-\frac{z}{t}}\right] + z^{i+1}t^2 \left[\frac{1-(tz)^{L-i+1}}{1-tz}\right] = \lambda z^i. \tag{5.9}$$

Collecting terms gives

$$t\left[\frac{t^{i-1}z}{1-\frac{z}{t}} - \frac{t^{L+2-i}z^{L+2}}{1-tz}\right] + tz^i\left[\frac{tz}{1-tz} - \frac{\frac{z}{t}}{1-\frac{z}{t}}\right] = \lambda z^i. \tag{5.10}$$

Since this needs to hold for all i, we obtain the eigenvalue λ as

$$\lambda = \lambda(t,z) = t\left[\frac{tz}{1-tz} - \frac{\frac{z}{t}}{1-\frac{z}{t}}\right] = -\frac{z(1-t^2)}{(1-tz)(1-\frac{z}{t})} = \frac{t(1-t^2)}{1-t(z+\frac{1}{z})+t^2}. \tag{5.11}$$

We immediately note that

$$\lambda(t,z) = \lambda(t,\tfrac{1}{z}) \tag{5.12}$$

and so to remove the boundary terms we extend the ansatz to

$$g_j = z^j + C(t,z)z^{-j}. \tag{5.13}$$

The same λ as above still works, and cancels the z^i and z^{-i} terms. We are left with the boundary equation

$$t\left[\frac{t^{i-1}z}{1-\frac{z}{t}} - \frac{t^{L+2-i}z^{L+2}}{1-tz}\right] + Ct\left[\frac{t^{i-1}}{z(1-\frac{1}{tz})} - \frac{t^{L+2-i}z^{-(L+2)}}{1-\frac{t}{z}}\right] = 0 \tag{5.14}$$

which after multiplying by t^{i-1} we rewrite as

$$\left[\frac{t^{2i-1}z}{1-\frac{z}{t}} - \frac{t^{L+2}z^{L+2}}{1-tz}\right] - C\left[\frac{t^{2i}}{1-tz} + \frac{t^{L+2}z^{-(L+2)}}{1-\frac{t}{z}}\right] = 0 \tag{5.15}$$

that is

$$t^{2i}\left[\frac{\frac{z}{t}}{1-\frac{z}{t}} - C\frac{1}{1-tz}\right] - t^{L+2}\left[\frac{z^{L+2}}{1-tz} + C\frac{z^{-(L+2)}}{1-\frac{t}{z}}\right] = 0. \tag{5.16}$$

This must hold for each i so we expect each term to be zero individually. We seek to set C so that there is a common factor between the two, which can then be cancelled by z. Comparing the two terms, we see that $C = \pm z^{L+2}$ will make them the same, up to a simple factor. First with $C = z^{L+2}$, the above becomes

$$t^{2i}\alpha(t,z) + t^{L+2}\alpha(t,z) = 0 \tag{5.17}$$

where

$$\alpha(t,z) = \frac{\frac{z}{t}}{1-\frac{z}{t}} - \frac{z^{L+2}}{1-zt} \tag{5.18}$$

$$= -\frac{1}{(1-tz)(1-\frac{t}{z})}\cdot(1-tz-tz^{L+1}+z^{L+2}). \tag{5.19}$$

On the other hand with $C = -z^{L+2}$, we get

$$t^{2i}\beta(t,z) - t^{L+2}\beta(t,z) = 0 \tag{5.20}$$

where

$$\beta(t,z) = \frac{\frac{z}{t}}{1-\frac{z}{t}} + \frac{z^{L+2}}{1-zt} \tag{5.21}$$

$$= -\frac{1}{(1-tz)(1-\frac{t}{z})}\cdot(1-tz+tz^{L+1}-z^{L+2}). \tag{5.22}$$

The above thus gives that the eigenvectors $\mathbf{g}_{L,k}$ of T_L, where $k = 1,\ldots,L+1$, are of the form

$$g_{L,k,j} = z^j + (-1)^{k+1}z^{L+2-j} \text{ with } j = 1,\ldots,L+1 \tag{5.23}$$

where the $z = z_{L,k}$ are complex numbers. Specifically, the $z_{L,k}$ are roots of the polynomials $A_{L,k}(t,z)$, which combine α and β from above:

$$A_{L,k}(t,z) = 1 - tz + (-1)^k z^{L+1}(t - z). \tag{5.24}$$

Note that

$$z^{L+2} A_{L,k}(t, \tfrac{1}{z}) = (-1)^{k+1} A_{L,k}(t,z) \tag{5.25}$$

so that if z is a root then so too is $\frac{1}{z}$. The property (5.25) makes $A_{L,k}$ a *self-inversive* polynomial, and in particular it is *palindromic* for odd k, and *antipalindromic* for even k. Since the roots come in reciprocal pairs, in the following $z_{L,k}$ can refer to either representative of a pair (it will make no difference which value is chosen).

Next, we observe that $A_{L,k}$ is of degree $L+2$, however

- when L,k are both odd, $A_{L,k}(t,-1) = 0$, but then at $z = -1$ we have $g_{L,k,j} = 0$ for all j,
- when L is odd and k is even, $A_{L,k}(t,1) = 0$, but then at $z = 1$ we again have $g_{L,k,j} = 0$,
- when L,k are both even, $A_{L,k}(t,1) = A_{L,k}(t,-1) = 0$, but then at $z = \pm 1$ we again have $g_{L,k,j} = 0$.

(Note that $A_{L,k}$ never has a double pole at $z = \pm 1$, which is easily seen by checking derivatives.) The roots at $z = \pm 1$ are thus trivial and are not counted among the $z_{L,k}$. Factoring out the trivial $(1 \pm z)$ terms then gives the polynomials

$$B_{L,k}(t,z) = 1 + (1+t)\sum_{n=1}^{L} (-1)^n z^n + z^{L+1} \qquad L,k \text{ odd} \tag{5.26}$$

$$B_{L,k}(t,z) = 1 + (1-t)\sum_{n=1}^{L} z^n + z^{L+1} \qquad L \text{ odd},\ k \text{ even} \tag{5.27}$$

$$B_{L,k}(t,z) = 1 - (t-z)\sum_{\substack{n=1 \\ n \text{ odd}}}^{L-1} z^n \qquad L,k \text{ even} \tag{5.28}$$

$$B_{L,k}(t,z) = 1 - tz - tz^{L+1} + z^{L+2} \qquad L \text{ even},\ k \text{ odd} \tag{5.29}$$

whose roots are exactly the reciprocal pairs $z_{L,k}$. It is easy to check that each of the $B_{L,k}(t,z)$ are palindromic. Indeed, setting $z = \frac{1}{z}$ in (5.23) leads to

$$z^{L+2} \left[\mathbf{g}_{L,k}\right]_{z=\frac{1}{z}} = (-1)^{k+1} \mathbf{g}_{L,k} \tag{5.30}$$

so that each reciprocal pair of roots gives the same eigenvalue/vector pair.

Next we diagonalise (using the fact that T_L is real symmetric), to get

$$G_L(t) = \frac{1}{t}(1,0,\ldots,0) \cdot \left(\sum_{k=1}^{L+1} \tilde{\mathbf{g}}_{L,k}^{\mathsf{T}} \lambda_{L,k}^{L+1} \tilde{\mathbf{g}}_{L,k} \right) \cdot (0,\ldots,0,1)^{\mathsf{T}} \tag{5.31}$$

where

$$\tilde{\mathbf{g}}_{L,k} = \frac{\mathbf{g}_{L,k}}{\|\mathbf{g}_{L,k}\|}. \tag{5.32}$$

Now

$$\|\mathbf{g}_{L,k}\|^2 = \sum_{j=1}^{L+1}(z_{L,k}^j + (-1)^{j+1} z_{L,k}^{L+2-j})^2 \tag{5.33}$$

$$= \frac{2z_{L,k}^2(1+(-1)^{k+1}(L+1)z_{L,k}^L(1-z_{L,k}^2) - z_{L,k}^{2L+2})}{1-z_{L,k}^2}. \tag{5.34}$$

Substituting,

$$P_L(t) = \frac{1}{t}(1,0,\ldots,0)\cdot\left(\sum_{k=1}^{L+1}\frac{\mathbf{g}_{L,k}^{\mathsf{T}}\lambda_{L,k}^{L+1}\mathbf{g}_{L,k}}{\|\mathbf{g}_{L,k}\|^2}\right)\cdot(0,\ldots,0,1)^{\mathsf{T}} \tag{5.35}$$

$$= \frac{1}{t}(1,0,\ldots,0)\cdot\left(\sum_{k=1}^{L+1}\mathbf{g}_{L,k}^{\mathsf{T}}\mathbf{g}_{L,k}\frac{(1-z_{L,k}^2)\lambda_{L,k}^{L+1}}{2z_{L,k}^2(1+(-1)^{k+1}(L+1)z_{L,k}^L(1-z_{L,k}^2)-z_{L,k}^{2L+2})}\right) \cdot(0,\ldots,0,1)^{\mathsf{T}} \tag{5.36}$$

$$= \frac{1}{t}\sum_{k=1}^{L+1}\frac{g_{L,k,1}g_{L,k,L+1}(1-z_{L,k}^2)\lambda_{L,k}^{L+1}}{2z_{L,k}^2(1+(-1)^{k+1}(L+1)z_{L,k}^L(1-z_{L,k}^2)-z_{L,k}^{2L+2})} \tag{5.37}$$

$$= \frac{(1-t^2)^{L+1}}{2t}\sum_{k=1}^{L+1}\frac{(-1)^{k+1}(1+(-1)^{k+1}z_{L,k}^L)^2(1-z_{L,k}^2)z_{L,k}^{L+1}}{(1+(-1)^{k+1}(L+1)z_{L,k}^L(1-z_{L,k}^2)-z_{L,k}^{2L+2})(z_{L,k}-t)^{L+1}(\frac{1}{t}-z_{L,k})^{L+1}}. \tag{5.38}$$

We again note that the above sum is over the $L+1$ reciprocal pairs of roots, and for each k it does not matter which of the pair is chosen. In the following subsection we will make things more explicit.

5.2. The roots for $t > 1$

The asymptotics of (5.38) depend on the values of the complex numbers $z_{L,k}$. There are $L+1$ (pairs) of these; however, it turns out that for $t > 1$ only two of them contribute to the dominant asymptotics. This is partly because of the following remarkable fact.

Lemma 2. *For $t > 1$ and $L > \frac{2}{t-1}$, $L-1$ of the reciprocal pairs of roots $z_{L,k}$ are on the unit circle, and two pairs (one for even k and one for odd k) are real, positive and not on the unit circle.*

See figure 5 for an illustration at $t = \frac{6}{5}$.

We will make use of a result due to Vieira. First, we make precise a term we used in the previous subsection. A polynomial

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n \tag{5.39}$$

with coefficients in $\mathbb{C}$ and with $a_n \neq 0$ is *self-inversive* if it satisfies

$$p(z) = \omega z^n \overline{p}(\tfrac{1}{z}) \tag{5.40}$$

with $|\omega| = 1$, where $\overline{p}(z)$ is the complex conjugate of $p(z)$.

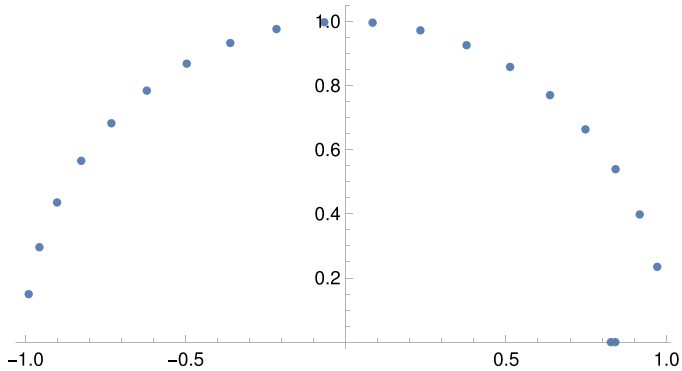


Figure 5. The roots $z_{20,k}$ in the complex plane at $t = \frac{6}{5} = 1.2$. For the reciprocal pairs on the unit circle we have chosen those with positive imaginary part, and for those on the real line we have chosen those inside the unit circle. Note the two real roots $z_{20,1}$ and $z_{20,2}$ close to $\frac{1}{t} = \frac{5}{6}$.

Lemma 3 ([25]). *Let $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ be a self-inversive polynomial of degree n. If*

$$|a_{n-l}| > \frac{1}{2} \sum_{\substack{k=0 \\ k \neq l, n-l}}^{n} |a_k|, \qquad l < \frac{n}{2}, \tag{5.41}$$

then p(z) has at least $n - 2l$ roots on the unit circle.

Proof of lemma 2. The polynomials $A_{L,k}(t,z)$ and $B_{L,k}(t,z)$ are self-inversive with $\omega = (-1)^{k+1}$ and $\omega = 1$ respectively. For now it is simpler to work with the $A_{L,k}$, keeping in mind the two trivial roots at $z = \pm 1$.

Take $p(z) = A_{L,k}(t,z)$ and set $l = 1$ in lemma 3. Then the condition (5.41) is simply $t > 1$, so at least L of the $L+2$ roots of $A_{L,k}$ are on the unit circle (note that these include the trivial roots), i.e. at most two are not on the unit circle. It remains to show that exactly two are not on the unit circle, both for odd and even k.

For odd k, any root satisfies

$$z^{L+1} = \frac{1 - tz}{t - z} = m(z). \tag{5.42}$$

For $z \in (0,1)$ we clearly have that z^{L+1} is a strictly increasing function $(0,1) \to (0,1)$. On the other hand

$$m'(z) = -\frac{t^2 - 1}{(t - z)^2} \tag{5.43}$$

so $m(z)$ is a strictly decreasing function mapping $(0,1)$ to $(\frac{1}{t}, -1)$. It follows that there must be a root $z = z_{L,1} \in (0,1)$. Since $A_{L,k}$ is self-inversive, there is another root at $z = \frac{1}{z_{L,1}} > 1$.

For even k, any root satisfies $z^{L+1} = -m(z)$. Now the RHS is a strictly increasing function $(0,1) \to (-\frac{1}{t}, 1)$. To establish the existence of a root, note that

$$\frac{\mathrm{d}}{\mathrm{d}z} z^{L+1} = (L+1) z^L \to (L+1) \text{ as } z \to 1^- \tag{5.44}$$

while

$$\frac{\mathrm{d}}{\mathrm{d}z}(-m(z)) = \frac{t^2-1}{(t-z)^2} \to \frac{t+1}{t-1} \text{ as } z \to 1^-. \tag{5.45}$$

Thus if

$$L+1 > \frac{t+1}{t-1} \qquad \Longleftrightarrow \qquad L > \frac{2}{t-1} \tag{5.46}$$

then as $z \to 1^-$, z^{L+1} approaches 1 at a greater slope than $-m(z)$, and hence $z^{L+1} < -m(z)$ for $z \in (1-\epsilon, 1)$ for some $\epsilon > 0$. So there is a real root $z = z_{L,2} \in (0,1)$. Again by the self-inversive property, there must be another at $z = \frac{1}{z_{L,2}} > 1$.

□

It is the two roots $z_{L,1}$ and $z_{L,2}$ inside the unit circle which are now of interest, and the next step is to compute the asymptotic behaviour of these as L grows large. First, observe that

$$A_{L,k}(t,\tfrac{1}{t}) = (-1)^k(\tfrac{1}{t})^{L+1}(t+\tfrac{1}{t}) \to 0 \qquad \text{as } L \to \infty, \tag{5.47}$$

while for $\varepsilon > 0$ with $0 < \frac{1}{t} - \epsilon < \frac{1}{t} + \epsilon < 1$ we have

$$A_{L,k}(t,\tfrac{1}{t} \pm \epsilon) \to \mp \epsilon t \qquad \text{as } L \to \infty. \tag{5.48}$$

It follows that the two roots $z_{L,1}$ and $z_{L,2}$ must approach $\frac{1}{t}$ as $L \to \infty$. Next, rearrange the equation $A_{L,k}(t,z) = 0$ to get

$$z^{L+1} = (-1)^{k+1}\frac{1-tz}{t-z}. \tag{5.49}$$

This implies that $z_{L,1} < \frac{1}{t}$ while $z_{L,2} > \frac{1}{t}$. Rearranging again,

$$\log\left[(-1)^{k+1}(\tfrac{1}{t}-z)\right] = (L+1)\log z + \log t + \log(t-z) \tag{5.50}$$

$$\sim -(L+1)\log t + \log t + \log(t - \tfrac{1}{t}) \tag{5.51}$$

$$= \log\left(\frac{t^2-1}{t^{L+3}}\right). \tag{5.52}$$

Hence for $k = 1,2$,

$$z_{L,k} \sim z^*_{L,k} = \frac{1}{t} + (-1)^k\left(\frac{t^2-1}{t^3}\right)t^{-L}. \tag{5.53}$$

It will turn out that the precision of these estimates is sufficient for even L but not enough for odd L (this is because there is significant cancellation between the $k=1$ and 2 terms of (5.38) for odd L). However, we can compute more a precise estimate for $z_{L,1}$ by iterating (5.50). That is, we substitute $z^*_{L,k}$ into the RHS of (5.50). Taking the next-to-leading term then gives

$$z_{L,k} \sim z^{**}_{L,k} = \frac{1}{t} + (-1)^k t^{-L}\left(\frac{t^2-1}{t^3}\right)\left(1 + (-1)^k t^{-L} L \frac{t^2-1}{t^2}\right). \tag{5.54}$$

5.3. Asymptotics

We will compute the leading asymptotics for $P_L(t)$ by taking only the $k=1$ and 2 terms from (5.38). We will then need to show that the remaining terms in the sum do not contribute to the dominant asymptotics, which amounts to showing that the first factor in the denominator of the summands is not too close to 0.

5.3.1. Even L. We take only the $k = 1, 2$ terms of (5.38). Any term of the form $z_{L,k}^L$ or similar approaches 0 very quickly, so for the purposes of asymptotics these are all set to 0, except for the factor of $z_{L,k}^{L+1}$ in the numerator. This, and the other $z_{L,k}$ terms except for the important $(\frac{1}{t} - z_{L,k})^{L+1}$ term in the denominator, are then set to $\frac{1}{t}$. This yields

$$\sim \frac{(1-t^2)^{2M+1}}{2t} \sum_{k=1}^{2} \frac{(-1)^{k+1}(1-\frac{1}{t^2})t^{-L-1}}{(\frac{1}{t}-t)^{L+1}(\frac{1}{t}-z_{L,k})^{L+1}} \tag{5.55}$$

$$= \frac{t^2-1}{2t^3}\left(\frac{1}{(\frac{1}{t}-z_{L,1})^{L+1}} - \frac{1}{(\frac{1}{t}-z_{L,2})^{L+1}}\right). \tag{5.56}$$

Now using the approximations $z_{L,k}^*$ this simplifies to

$$\sim \left(\frac{t^4}{t^2-1}\right)^L t^{L^2}, \qquad L \text{ even}. \tag{5.57}$$

5.3.2. Odd L. If we follow the same procedure as above but take L to be odd then everything cancels and we just get 0. So we must instead switch to the more precise estimates $z_{L,k}^{**}$. Substituting, we get

$$\sim \frac{t^{2L^2+7L+2}}{2(t^2-1)^L}\left(\frac{1}{(t^{L+2}-L(t^2-1))^{L+1}} - \frac{1}{(t^{L+2}+L(t^2-1))^{L+1}}\right) \tag{5.58}$$

$$\sim \frac{t^{2L^2+7L+2}}{2(t^2-1)^L} \cdot \frac{1}{t^{L(L+2)}}\left(\frac{1}{t^{L+2}-L(L+1)(t^2-1)} - \frac{1}{t^{L+2}+L(L+1)(t^2-1)}\right) \tag{5.59}$$

$$= \frac{t^{L^2+3L-2}}{(t^2-1)^L} \cdot \frac{L(L+1)(t^2-1)}{1-\frac{L^2(L+1)^2(t^2-1)^2}{t^{2L+4}}} \tag{5.60}$$

$$\sim \frac{t^2-1}{t^2} \cdot L^2 \cdot \left(\frac{t^3}{t^2-1}\right)^L \cdot t^{L^2}, \qquad L \text{ odd}. \tag{5.61}$$

To get from the first to the second line above we have used $(1+x)^L \sim 1 + Lx$ for each of the two terms in the large parentheses.

5.3.3. The roots on the unit circle. With factors of the form t^{L^2} coming from the $k = 1$ and 2 terms in the sum (5.38), the remaining terms can only affect the dominant asymptotics if the factor

$$D_{L,k}(t) = 1 + (-1)^{k+1}(L+1)z_{L,k}^L(1-z_{L,k}^2) - z_{L,k}^{2L+2} \tag{5.62}$$

in the denominator is very close to 0. Here we show this is not the case. Firstly, (5.49) can be used to eliminate the $z_{L,k}^L$ and $z_{L,k}^{2L+2}$ terms, giving

$$D_{L,k}(t) = 1 + (L+1)(1-z_{L,k}^2)\frac{1-tz_{L,k}}{z_{L,k}(t-z_{L,k})} - \frac{(1-tz_{L,k})^2}{(t-z_{L,k})^2}. \tag{5.63}$$

Since the $z_{L,k}$ are all on the unit circle and $0 < t < 1$, the asymptotics of this (for large L) are

$$D_{L,k}(t) = \frac{(1-tz_{L,k})(1-z_{L,k}^2)}{z_{L_k}(t-z_{L,k})}L + O(1). \tag{5.64}$$

Now

$$m(z) = \frac{1 - tz}{t - z} \tag{5.65}$$

is a Möbius transformation which maps the unit circle to itself. Hence for $z = e^{i\theta}$ on the unit circle,

$$\left| \frac{(1 - tz)(1 - z^2)}{z(t - z)} \right| = \left| \frac{1 - z^2}{z} \right| = 2|\sin\theta|. \tag{5.66}$$

Let us assume we choose all the $z_{L,k}$ to be in the upper half unit circle. Then if we can show that none of the roots are too close to ± 1, i.e. there are no roots of the form $z = e^{i\theta}$ with θ close to 0 or π, then $|D_{L,k}|$ cannot be very small. We will show that if $z_{L,k} = e^{i\theta}$ with $0 < \theta < \pi$, then in fact

$$\frac{\pi}{L+1} \leqslant \theta \leqslant \pi - \frac{\pi}{L+1}, \tag{5.67}$$

from which it follows that

$$|D_{L,k}(t)| \geqslant 2\pi + O(L^{-1}). \tag{5.68}$$

(This bound is in fact tight—if we order the roots for $k = 3, \ldots, L+1$ anticlockwise from right to left, then at $k = 3$ and $k = L+1$ we have $|D_{L,k}(t)| \to 2\pi$ as $L \to \infty$, for all $t > 1$. We will make no attempt to prove this here, however.)

We wish to show that if $z = e^{i\theta}$ with $0 < \theta < \frac{\pi}{L+1}$ or $\pi - \frac{\pi}{L+1} < \theta < \pi$, then z cannot be a root of $A_{L,k}(t, z) = 0$. There are a number of cases, which we will briefly consider in turn.

*5.3.3.1. **I.** Odd k, small θ.* We have $z^{L+1} = m(z)$. If $0 < \theta < \frac{\pi}{L+1}$ then the LHS will be in the upper half of the unit circle. The RHS is a Möbius transformation which maps the upper half of the unit circle to the lower half, so there is no root.

*5.3.3.2. **II.** Odd k, even L, large θ.* Write $\theta = \pi - \phi$. Then

$$z^{L+1} = e^{i\theta(L+1)} = e^{i\pi(L+1)} e^{-i\phi(L+1)} = -e^{-i\phi(L+1)} \tag{5.69}$$

which is again in the upper half of the unit circle.

*5.3.3.3. **III.** Odd k, odd L, large θ.* This time $z^{L+1} = e^{-i\phi(L+1)}$ which is in the lower half of the unit circle. However, observe that as $\phi \nearrow \frac{\pi}{L+1}$ we have

$$\arg(z^{L+1}) \searrow -\pi < \arg(m(z)) \tag{5.70}$$

(where we take arguments to be between $-\pi$ and π). Then since

$$\frac{d}{d\theta} \arg(m(e^{i\theta})) = \frac{t^2 - 1}{t^2 + 1 - 2t\cos\theta} < 1 \tag{5.71}$$

while

$$\frac{d}{d\theta} \arg(e^{i\theta(L+1)}) = L + 1, \tag{5.72}$$

we must have $\arg(z^{L+1}) < \arg(m(z))$, so there can be no root for $\pi - \frac{\pi}{L+1} < \theta < \pi$.

*5.3.3.4. **IV.** Even k, odd L, large θ.* With even k we have $z^{L+1} = -m(z)$. The RHS is now a Möbius transformation mapping the upper half of the unit circle to itself. The rest of this case is analogous to case **II** above.

*5.3.3.5. **V.** Even k, small θ.* This uses the same argument as case **III** above—one shows that $\arg(z^{L+1}) > \arg(-m(z))$.

*5.3.3.6. **VI.** Even k, even L, large θ.* Similar to cases **III** and **V** above, this time showing that $\arg(z^{L+1}) < \arg(-m(z))$.

6. Discussion

We have examined a model of partially directed walks crossing an $L \times L$ square, weighted by a fugacity t according to their length. We have obtained two different solutions for the partition functions, using generating functions and the kernel method as well as transfer matrices and a Bethe ansatz. The former allowed the calculation of the dominant asymptotics for the dilute $t < 1$ regime and the latter worked for the dense $t > 1$ regime. As such we provide an exact solution of a simplified model of the dilute to dense polymeric phase transition.

6.1. Comparison with SAWs

In section 1 we discussed what is known or believed about the model of SAWs crossing a box. Some of these results can now be contrasted with those for PDWs. For example we have

$$f_1(t) = \log(4t^2) - \log(1-t^2) \qquad \text{and} \qquad f_2(t) = \log t. \tag{6.1}$$

Thus in the dilute phase we do not have power-law divergence as $t \to 1^-$: instead $f_1(t)$ diverges only logarithmically. On the other hand

$$f_2(t) \sim (t-1), \qquad t \to 1^+. \tag{6.2}$$

At the critical point things are also different—SAWs are expected to feature power-law behaviour as per (1.7), but PDWs instead have an $L\log L$ in the exponent as in (2.4).

For the asymptotics of the mean length $\langle n\rangle_L(t)$, away from the critical point PDWs and SAWs both exhibit the same behaviour as per (1.12). Note that for PDWs the coefficient of L for $t < 1$ diverges as $t \to 1^-$ (see (2.8)), and the same would be expected for SAWs. At the critical point SAWs are expected to have $L^{1/\nu}$ asymptotics while PDWs have L^2.

We observe here that PDWs, having a 'preferred' direction, have two different values of ν—namely $\nu_{\|} = 1$ in the preferred direction and $\nu_{\perp} = \frac{1}{2}$ in the perpendicular direction [24]. Using the exponent $\nu_{\perp}$ we see that PDWs then match the expected behaviour for SAWs both in the scaling of f_2 near criticality (see (1.10) and (6.2)) and the asymptotics of $\langle n\rangle_L(\mu^{-1})$.

6.2. Other models

There are a number of other possible models of confined polymers which may be worth considering. It may be natural to consider random walks, that is, walks which lack any self-avoidance constraint. However great care must be taken, because in that case the partition functions as in (1.1) are no longer polynomials, but instead power series (because there are infinitely many random walks between two points in the box). There would then be values of t for which these series diverge and things are not well defined. Whether phase transitions can still be observed is an open question.

Interpolating between random walks and SAWs are *weakly self-avoiding* walks (see e.g. [1]), which are weighted according to the number of vertices visited multiple times.

Some of the results and conjectures for SAWs apply in higher dimensions (e.g. [18]). There are multiple ways to define PDWs in more than two dimensions (one has to choose how many dimensions in which the walks are directed), but at least some partial results are probably attainable.

There are also ways to change the model by varying the boundary conditions. This can involve relaxing the restrictions on where the walks start and end—for example they can be restricted to end on opposing edges instead of corners (e.g. [2]) or to simply be contained within the square [3]. The boundaries themselves may also be periodic. See also [29] and references therein for some recent work regarding SAWs and random walks in boxes, with free and periodic boundary conditions, and their connections with the Ising model.

Finally, we remind the reader that the scaling limit of SAWs crossing a box is conjectured to be the Schramm-Loewner evolution (SLE) with parameter $\kappa = \frac{8}{3}$ (specifically the chordal version of SLE, see e.g. [16]). The scaling limit for PDWs has been studied recently (see e.g. [7, 17], including results for PDWs with nearest-neighbour interactions). In the unweighted case the scaling limit will be some version of Brownian motion; for $t \neq 1$, however, it remains an open question.

Data availability statement

No new data were created or analysed in this study.

ORCID iDs

Nicholas R Beaton https://orcid.org/0000-0001-8220-3917
Aleksander L Owczarek https://orcid.org/0000-0001-8919-3424

References

[1] Bauerschmidt R, Brydges D C and Slade G 2015 Critical two-point function of the 4-dimensional weakly self-avoiding walk *Commun. Math. Phys.* **338** 169–93
[2] Bousquet-Mélou M, Guttmann A J and Jensen I 2005 Self-avoiding walks crossing a square *J. Phys. A: Math. Gen.* **38** 9159–81
[3] Bradly C J and Owczarek A L 2022 Critical scaling of lattice polymers confined to a box without endpoint restriction *J. Math. Chem.* **60** 1903–20
[4] Brak R, Dyke P, Lee J, Owczarek A L, Prellberg T, Rechnitzer A and Whittington S G 2009 A self-interacting partially directed walk subject to a force *J. Phys. A: Math. Theor.* **42** 085001
[5] Burkhardt T W and Guim I 1991 Self-avoiding walks that cross a square *J. Phys. A: Math. Gen.* **24** L1221
[6] Cardy J L 1984 Conformal invariance and surface critical behavior *Nucl. Phys.* B **240** 514–32
[7] Carmona P and Pétrélis N 2016 Interacting partially directed self avoiding walk: scaling limits *Electron. J. Probab.* **21** 1–52
[8] Forgacs G 1991 Unbinding of semiflexible directed polymers in 1+1 dimensions *J. Phys. A: Math. Theor.* **24** L1099
[9] Guttmann A J and Jensen I 2022 Self-avoiding walks and polygons crossing a domain on the square and hexagonal lattices *J. Phys. A: Math. Theor.* **55** 485201
[10] Guttmann A J, Jensen I and Owczarek A L 2022 Self-avoiding walks contained within a square *J. Phys. A: Math. Theor.* **55** 425201

[11] Janse van Rensburg E J and Prellberg T 2016 Forces and pressures in adsorbing partially directed walks *J. Phys. A: Math. Theor.* **49** 205001
[12] Janse van Rensburg E J, Prellberg T and Rechnitzer A 2008 Partially directed paths in a wedge *J. Comb. Theory* A **115** 623–50
[13] Knuth D E 1976 Mathematics and computer science: coping with finiteness *Science* **194** 1235–42
[14] Knuth D E 1997 *The Art of Computer Programming, Volume 1: Fundamental Algorithms* 3rd edn (Reading, MA: Addison-Wesley)
[15] Lam P-M, Zhen Y, Zhou H and Zhou J 2009 Adsorption of externally stretched two-dimensional flexible and semiflexible polymers near an attractive wall *Phys. Rev.* E **79** 061127
[16] Lawler G F, Schramm O and Werner W 2004 On the scaling limit of planar self-avoiding walk *Fractal Geometry and Applications: A Jubilee of Benoît Mandelbrot. Multifractals, Probability and Statistical Mechanics, Applications. In Part the Proceedings of a Special Session Held During the Annual Meeting of the American Mathematical Society, San Diego, CA, USA, January 2002* (Providence, RI: American Mathematical Society) pp 339–64
[17] Legrand A and Pétrélis N 2022 A sharp asymptotics of the partition function for the collapsed interacting partially directed self-avoiding walk *J. Stat. Phys.* **186** 41
[18] Madras N 1995 Critical behaviour of self-avoiding walks that cross a square *J. Phys. A: Math. Gen.* **28** 1535
[19] Madras N and Slade G 1996 *The Self-Avoiding Walk* (Boston, MA: Birkhäuser)
[20] Owczarek A L 2009 Exact solution for semi-flexible partially directed walks at an adsorbing wall *J. Stat. Mech.* P11002
[21] Owczarek A L 2010 Effect of stiffness on the pulling of an adsorbing polymer from a wall: an exact solution of a partially directed walk model *J. Phys. A: Math. Theor.* **43** 225002
[22] Owczarek A L, Brak R and Rechnitzer A 2009 Self-avoiding walks in slits and slabs with interactive walls *J. Math. Chem.* **45** 113–28
[23] Owczarek A L and Prellberg T 2007 Exact solution of semi-flexible and super-flexible interacting partially directed walks *J. Stat. Mech.* P11010
[24] Owczarek A L, Prellberg T and Brak R 1993 The tricritical behaviour of self-interacting partially directed walks *J. Stat. Phys.* **72** 737–72
[25] Vieira R S 2017 On the number of roots of self-inversive polynomials on the complex unit circle *Ramanujan J.* **42** 363–9
[26] Whittington S G 2022 Self-avoiding walks and polygons confined to a square (arXiv:2211.16287)
[27] Whittington S G and Guttmann A J 1990 Self-avoiding walks which cross a square *J. Phys. A: Math. Gen.* **23** 5601–9
[28] Zhou H, Zhou J, Ou-Yang Z-C and Kumar S 2006 Collapse transition of two-dimensional flexible and semiflexible polymers *Phys. Rev. Lett.* **97** 158302
[29] Zhou Z, Grimm J, Fang S, Deng Y and Garoni T M 2018 Random-length random walks and finite-size scaling in high dimensions *Phys. Rev. Lett.* **121** 185701