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J. Stat. Mech. (2012) P11001

(http://iopscience.iop.org/1742-5468/2012/11/P11001)

Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Directed compact percolation near a damp wall with biased growth

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Received 18 September 2012
Accepted 11 October 2012
Published 2 November 2012

Online at stacks.iop.org/JSTAT/2012/P11001
doi:10.1088/1742-5468/2012/11/P11001

Abstract. The model of directed compact percolation near a damp wall is generalized to allow for a bias in the growth of a cluster, either towards or away from the wall. The percolation probability for clusters beginning with seed width m, any distance from the wall, is derived exactly by solving the associated recurrences. It is found that the general biased case near a damp wall leads to a critical exponent $\beta = 1$, in line with the dry biased case, which differs from the unbiased damp/dry exponent $\beta = 2$.

Keywords: solvable lattice models, percolation problems (theory)

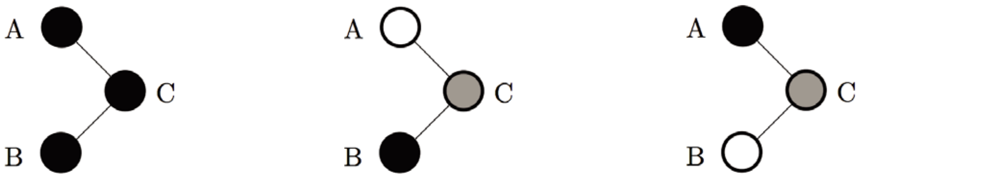
 1742-5468/12/P11001+18$33.00

Contents

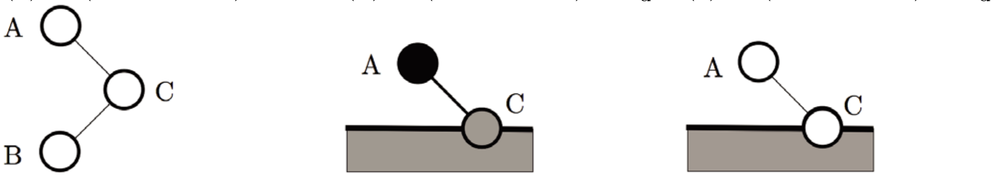
1. Introduction

Lattice models of percolation have continued to play a central role in our understanding of the integrability and critical behaviour of statistical mechanical systems [1]–[3]. They also model a vast array of physical systems [4]–[7]. One of the integrable models of percolation is that of directed compact percolation. The directed compact percolation model, first introduced by Domany and Kinzel [8], has been modified in several ways to investigate this exactly solvable model. Essam [9] generalized the bulk case to allow for a bias in the

Table 1. The rules of the growth process for directed compact percolation near a damp wall, with bias.



(a) Pr(C occupied) $= 1$ (b) Pr(C occupied) $= p_{\mathrm{u}}$ (c) Pr(C occupied) $= p_{\mathrm{d}}$



(d) Pr(C occupied) $= 0$ (e) Pr(C occupied) $= p_{\mathrm{w}}$ (f) Pr(C occupied) $= 0$

growth of clusters, which did not change the critical behaviour of the cluster properties. Another modification that has been considered in several different cases is the addition of a wall to restrict the lateral growth of a cluster. A wet wall causes the cluster to remain attached to the wall, and was shown in [10] to have the same critical behaviour as the bulk case. This model naturally exhibits a pre-determined biased growth, since the attachment to the wall means there is a strong bias towards the wall, with the cluster moving towards the wall with certainty, leaving the probability away from the wall as the only free variable.

The addition of a *dry* or non-conducting wall, considered in [11]–[14], produced different critical exponents from the wet and bulk cases. However, it was found in [10] that when a bias is introduced, either towards or away from the dry wall, the exponents revert to the bulk/wet values; so it is only the unbiased dry case which exhibits different critical behaviour. Due to the differences between the wet and dry wall models, it was of interest to consider a *damp* wall which interpolates between the two. The unbiased case of the damp wall model has been considered in [15]–[18], and was found to produce the same critical behaviour as the unbiased dry wall model. This paper extends the work in the damp case to consider biased growth, building on the equivalent work which has been done for a dry wall, and it is found that again it is only the unbiased case which exhibits different critical behaviour from the bulk, with any bias resulting in the bulk/wet critical exponents.

Thus we adapt the model of directed compact percolation near a damp wall [15] to introduce a bias in the growth of clusters, as was done for the dry wall in [10]. The model is defined on a directed square lattice, the sites of which are the points in the t, x plane with integer coordinates such that $t \geq 0$, $x \geq 1$ and $t + x$ is even. The damp wall is represented by the sites at $x = 1$, where each wall site is 'wet' (occupied) with probability p_{w} and 'dry' (unoccupied) with probability $q_{\mathrm{w}} = 1 - p_{\mathrm{w}}$. We begin with an initial *seed* of m contiguous sites at $t = 0$, the midpoint of which is located y units above the wall. The seed is placed with certainty, and a cluster is grown from this one column at a time, according to the rules of directed compact percolation as illustrated in table 1.

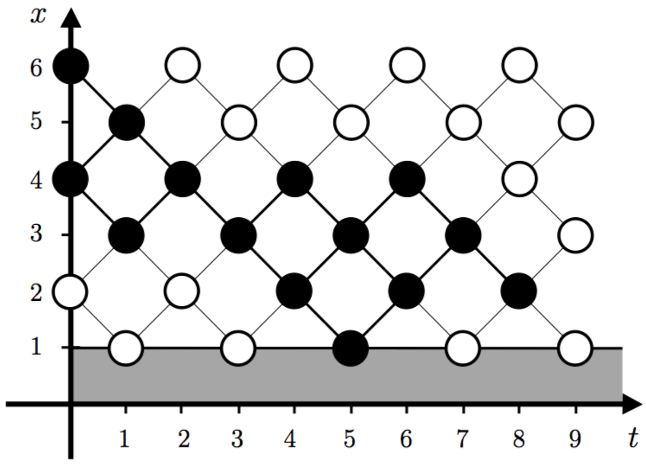


Figure 1. An example cluster, grown from a seed of width $m = 2$ with midpoint located $y = 4$ units from the wall. The probability of this cluster being grown from the seed can be calculated a column at a time, as $(1)(q_\mathrm{u}p_\mathrm{d})(q_\mathrm{u}q_\mathrm{d})(q_\mathrm{u}p_\mathrm{d})(p_\mathrm{u}p_\mathrm{d})(q_\mathrm{u}p_\mathrm{w})(p_\mathrm{u})(q_\mathrm{u}q_\mathrm{w})(q_\mathrm{u}p_\mathrm{d})(q_\mathrm{u}q_\mathrm{w}) = p_\mathrm{u}^2 q_\mathrm{u}^7 p_\mathrm{d}^4 q_\mathrm{d} p_\mathrm{w} q_\mathrm{w}^2$.

To ensure compactness, the site (t, x) becomes wet with certainty if the sites $(t-1, x \pm 1)$ are both wet. If only $(t-1, x-1)$ is wet then the site (t, x) becomes wet with probability p_u, corresponding to cluster growth in the 'upward' direction, and hence remains dry with probability $q_\mathrm{u} = 1 - p_\mathrm{u}$. Similarly if only $(t-1, x+1)$ is wet then the site (t, x) becomes wet with probability p_d, corresponding to cluster growth in the 'downward' direction, and remains dry with probability $q_\mathrm{d} = 1 - p_\mathrm{d}$. Where the sites $(t-1, x \pm 1)$ are both dry, the site (t, x) remains dry with certainty. In this way each successive column can be determined, and then acts as a seed for the remainder of the cluster. We can similarly use these rules to calculate the probability that a particular cluster is grown from a given seed by this growth process, as in the example in figure 1, or to consider all possible clusters that can be grown from a given seed.

We define the percolation probability $P_{m,y}(p_\mathrm{u}, p_\mathrm{d}, p_\mathrm{w})$ to be the probability that a cluster grown from a given seed, of width m and midpoint y units from a damp wall, becomes an infinite cluster. Within the range of possible values of the probabilities p_u, p_d and p_w, there will be some *low-density region*, where there are only finite clusters, for which the percolation probability will be zero. Above some critical threshold, determined by these probabilities, there will be a high-density region where there is a non-zero probability of a given seed producing an infinite cluster. That is, we have

$$P_{m,y}(p_\mathrm{u}, p_\mathrm{d}, p_\mathrm{w}) = \begin{cases} 0, & \text{low-density region;} \\ P_{m,y}, & \text{high-density region;} \end{cases} \tag{1}$$

where $P_{m,y} > 0$.

The behaviour of the percolation probability approaching this critical threshold from above is given by a simple power law, with critical exponent β. In the bulk [9] and wet wall [10] cases, it was found that $\beta = 1$, whereas the dry wall case [10] was found to have critical exponent $\beta = 2$ in the unbiased case, reverting to $\beta = 1$ when a bias towards or away from the wall was introduced. The specific unbiased damp case considered in [15] was similarly found to have $\beta = 2$, which we will confirm for clusters grown from seeds of any width or distance from the wall.

In this paper we find the percolation probability, and associated critical exponent, in the biased damp case. Despite the addition of an extra variable, $p_{\rm w}$, we find that the percolation probability for the biased case of directed compact percolation near a damp wall can be found using similar methods to the dry case. By solving the recurrence relations, we arrive at a general expression for percolation probability near a damp wall under biased growth, for clusters beginning with a seed of width m with midpoint y units from the wall:

$$P_{m,y} = 1 - \left(\frac{q_{\rm u}q_{\rm d}}{p_{\rm u}p_{\rm d}}\right)^m - \frac{(p_{\rm u}-q_{\rm d})(p_{\rm u}p_{\rm w}-p_{\rm d})}{(p_{\rm u}-p_{\rm d})(p_{\rm d}-p_{\rm w}q_{\rm u})}\left(\left(\frac{q_{\rm u}}{p_{\rm u}}\right)^m - \left(\frac{q_{\rm d}}{p_{\rm d}}\right)^m\right)\left(\frac{q_{\rm u}}{p_{\rm u}}\right)^y, \qquad y \geq m, \tag{2}$$

$$P_{m,m-1} = 1 - \frac{q_{\rm u}q_{\rm w}(2p_{\rm u}-1)}{(p_{\rm u}-p_{\rm d})(p_{\rm d}-p_{\rm w}q_{\rm u})}\left(\frac{q_{\rm u}q_{\rm d}}{p_{\rm u}p_{\rm d}}\right)^{m-1} - \frac{(p_{\rm u}-q_{\rm d})(p_{\rm u}p_{\rm w}-p_{\rm d})}{(p_{\rm u}-p_{\rm d})(p_{\rm d}-p_{\rm w}q_{\rm u})}\left(\frac{q_{\rm u}}{p_{\rm u}}\right)^{2m-1}, \tag{3}$$

which has critical exponent $\beta = 1$, when $p_{\rm u} \neq p_{\rm d}$, in line with the bulk, wet and biased dry wall results. However, the unbiased damp case of $p_{\rm u} = p_{\rm d}$ leads to the different critical exponent $\beta = 2$, generalizing the result found in [15] to general cluster size and position, as in the unbiased dry case.

2. Calculation

2.1. Recurrences for $r_t(m, y)$

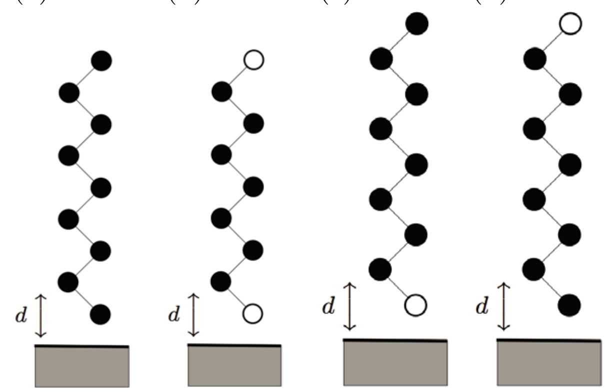
Let $r_t(m, y)$ be the probability that a cluster grown from a seed of width m, midpoint y units from the wall, has exactly t growth stages before terminating. We now set up recurrences for $r_t(m, y)$ by considering the growth of the cluster from one time step to the next. As this will differ depending on the interaction with the wall, we treat separately three classifications of seed location: away from the wall, adjacent to the wall or on the wall.

2.1.1. Away from the wall, $y > m$. A seed with lowest occupied site a distance $d \geq 2$ from the wall will not lead to any interaction with the wall in the following column. This corresponds to the midpoint of the seed being a distance $y > m$ from the wall, which gives four possibilities for cluster growth in the following time step, as shown in table 2. When the cluster's midpoint, or centre of mass, shifts upwards, this corresponds to the 'top' of the cluster propagating upwards with probability $p_{\rm u}$ and the site adjacent to the bottom of the cluster remaining unoccupied with probability $q_{\rm d}$. Similarly shifting downwards corresponds to an unoccupied 'up' site with probability $q_{\rm u}$, and the cluster propagating downward with probability $p_{\rm d}$. When the cluster width increases or decreases, it means both the 'up' and 'down' adjacent sites have either become occupied with probability $p_{\rm u}p_{\rm d}$, causing the width to increase, or remained unoccupied with probability $q_{\rm u}q_{\rm d}$, causing the width to decrease. We can write this as the recurrence

$$\begin{aligned} r_t(m,y) = {} & p_{\rm u}q_{\rm d}r_{t-1}(m,y+1) + q_{\rm u}p_{\rm d}r_{t-1}(m,y-1) + p_{\rm u}p_{\rm d}r_{t-1}(m+1,y) \\ & + q_{\rm u}q_{\rm d}r_{t-1}(m-1,y), \qquad y > m, \quad m > 1, \quad t > 0. \end{aligned} \tag{4}$$

Table 2. All possible configurations, and corresponding probabilities, for clusters beginning distance $d \geq 2$ away from the wall, illustrated with a cluster of initial seed width $m = 4$.

Possible configurations:	(a)	(b)	(c)	(d)



Second column, $t = 1$				
Cluster width:	$m+1$	$m-1$	m	m
Midpoint–wall distance:	y	y	$y+1$	$y-1$
Probability:	$p_\mathrm{u}p_\mathrm{d}$	$q_\mathrm{u}q_\mathrm{d}$	$p_\mathrm{u}q_\mathrm{d}$	$q_\mathrm{u}p_\mathrm{d}$

We consider separately the case $m = 1$, for which one of the configurations will cause the cluster to terminate, hence we have

$$r_t(1, y) = p_\mathrm{u}q_\mathrm{d}r_{t-1}(1, y+1) + q_\mathrm{u}p_\mathrm{d}r_{t-1}(1, y-1) + p_\mathrm{u}p_\mathrm{d}r_{t-1}(2, y), \qquad y > 1, \quad t > 0, \tag{5}$$

where this case can be covered by (4) if we define

$$r_t(0, y) = 0, \qquad y > m, \quad t > 0. \tag{6}$$

We now consider the case $t = 0$. For $r_0(m, y)$ to be non-zero, it must be possible for the cluster to terminate immediately. Due to the rules of compactness, a cluster with seed width $m > 1$ must propagate into the next column, so cannot terminate immediately. So we have

$$r_0(m, y) = 0, \qquad m \geq 2, \quad y > m. \tag{7}$$

However, a seed consisting of a single occupied site has some probability of terminating immediately, if both adjacent sites remain unoccupied. Away from the wall this probability is determined by the two bulk occupation probabilities p_u and p_d, so for the cluster to terminate we have

$$r_0(1, y) = q_\mathrm{u}q_\mathrm{d}, \qquad y > 1. \tag{8}$$

2.1.2. Adjacent to the wall, $y = m$. We say that a seed is *adjacent* to the wall if its lowest site is immediately above the wall, and as such we must consider the occupancy of a wall site in the next time step. This corresponds to a seed with midpoint $y = m$ units from the

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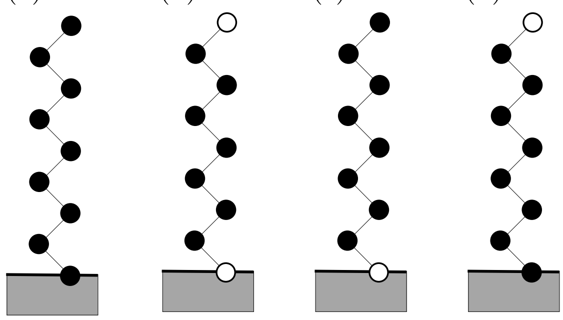
Table 3. The different configurations possible for a cluster beginning *adjacent* to the wall, and their probabilities, shown through a sample cluster of initial seed width $m = 4$.

Possible configurations:	(a)	(b)	(c)	(d)
				
Second column, $t = 1$				
Cluster width:	$m + 1$	$m - 1$	m	m
Midpoint–wall distance:	m	m	$m + 1$	$m - 1$
Probability:	$p_{\mathrm{u}}p_q$	$q_{\mathrm{u}}q_{\mathrm{w}}$	$p_{\mathrm{u}}q_{\mathrm{w}}$	$q_{\mathrm{u}}p_{\mathrm{w}}$

wall, and leads to the same four possibilities of physical configurations as in the bulk case, as seen in table 3. We adjust the probability of each configuration compared to the bulk case by simply replacing p_{d} with p_{w} in the recurrence in (4), as downward movement of the cluster will correspond to an occupied wall site. This results in the recurrence adjacent to the wall:

$$r_t(m,m) = p_{\mathrm{u}}q_{\mathrm{w}}r_{t-1}(m,m+1) + q_{\mathrm{u}}p_{\mathrm{w}}r_{t-1}(m,m-1) \\ + p_{\mathrm{u}}p_{\mathrm{w}}r_{t-1}(m+1,m) + q_{\mathrm{u}}q_{\mathrm{w}}r_{t-1}(m-1,m), \qquad m > 1, \quad t > 0. \tag{9}$$

For the case $m = 1$ we have a similar relationship, omitting only the final term which would correspond to a terminated cluster. Thus we have

$$r_t(1,1) = p_{\mathrm{u}}q_{\mathrm{w}}r_{t-1}(1,2) + q_{\mathrm{u}}p_{\mathrm{w}}r_{t-1}(1,0) + p_{\mathrm{u}}p_{\mathrm{w}}r_{t-1}(2,1), \qquad t > 0, \tag{10}$$

which can be united with (9) by the definition

$$r_t(0,1) = 0. \tag{11}$$

The case $t = 0$ corresponds to a cluster terminating immediately, which is only possible for a seed of width 1, so for all other seeds we have

$$r_0(m,m) = 0, \qquad m > 1. \tag{12}$$

A single site adjacent to the wall will terminate if both the wall site is dry, with probability q_{w}, and the cluster does not propagate upwards, with probability q_{u}, hence

$$r_0(1,1) = q_{\mathrm{u}}q_{\mathrm{w}}. \tag{13}$$

2.1.3. On the wall, $y = m - 1$. When $y = m - 1$, the seed includes a site on the wall. In this case there can be no further downward movement of the cluster in the next time step, and there are only two possibilities—as shown in table 4. These depend solely on

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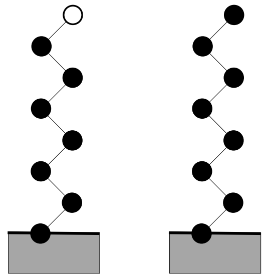
Table 4. The different configurations possible for a cluster beginning *on* the wall, and their probabilities, shown through a sample cluster of initial seed width $m = 4$.

Possible configurations:	(a)	(b)



Second column, $t = 1$		
Cluster width:	$m - 1$	m
Midpoint–wall distance:	$m - 1$	m
Probability:	q_{u}	p_{u}

the occupation of a single site adjacent to the seed in the 'up' direction. So we have the recurrence

$$r_t(m, m-1) = p_{\mathrm{u}} r_{t-1}(m, m) + q_{\mathrm{u}} r_{t-1}(m-1, m-1), \qquad m \geq 1, \quad t > 0. \tag{14}$$

Again we consider separately the $m = 1$ case, which eliminates one of the possible configurations with the result

$$r_t(1, 0) = p_{\mathrm{u}} r_{t-1}(1, 1), \qquad t > 0, \tag{15}$$

which can be covered by (14) if we make the definition

$$r_t(0, 0) = 0. \tag{16}$$

The case $t = 0$ corresponds to a cluster terminating immediately, which is not possible for clusters with seed width $m > 1$, so we have

$$r_0(m, m-1) = 0, \qquad m > 1. \tag{17}$$

We note that a single site on the wall can only propagate upwards, and will hence terminate with probability q_{u}, giving

$$r_0(1, 0) = q_{\mathrm{u}}. \tag{18}$$

2.2. Solving for $Q(m, y)$

We define $Q(m, y)$ to be the probability that a finite cluster is grown from a seed of width m and midpoint y units from the wall. This can be expressed as the sum of all finite clusters of length t grown from that seed, so we can define

$$Q(m, y) = \sum_{t=0}^{\infty} r_t(m, y), \qquad m > 0, \quad y \geq m - 1, \tag{19}$$

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noting that we have omitted the trivial case of $m = 0$ from our definition. Although we have defined $r_t(0, y)$, this was for convenience—so that the more general recurrence would hold for the $m = 1$ case for each seed classification—and did not come from a physical interpretation. We note that $Q(m, y)$ will be of the form

$$Q(m,y) = \begin{cases} 1, & \text{low-density region;} \\ Q_{m,y}, & \text{high-density/percolating region.} \end{cases} \tag{20}$$

Thus, in finding the expression in the high-density region, $Q_{m,y}$, we will also discover the percolating region for the biased damp case. We will then be able to find an expression for the percolation probability in the biased damp case, which in the high-density region is equal to

$$P_{m,y} = 1 - Q_{m,y}. \tag{21}$$

2.2.1. Recurrences for $Q_{m,y}$. We can obtain recurrences for $Q_{m,y}$ by summing those found for $r_t(m, y)$ over $t \geq 1$ and adjusting for the $t = 0$ term where necessary to obtain the form in (19). Away from the wall, that is $y > m$, we obtain a general recurrence for $Q_{m,y}$ by summing (4) for $t \geq 1$. We treat the $m = 1$ case separately, as this is the only recurrence which will lead to a non-zero $t = 0$ term as given in (8). The resultant recurrences for $Q_{m,y}$ in the bulk are

$$Q_{m,y} = p_{\mathrm{u}}q_{\mathrm{d}}Q_{m,y+1} + q_{\mathrm{u}}p_{\mathrm{d}}Q_{m,y-1} + p_{\mathrm{u}}p_{\mathrm{d}}Q_{m+1,y} + q_{\mathrm{u}}q_{\mathrm{d}}Q_{m-1,y}, \qquad m > 1, \quad y > m, \tag{22}$$

$$Q_{1,y} = p_{\mathrm{u}}q_{\mathrm{d}}Q_{1,y+1} + q_{\mathrm{u}}p_{\mathrm{d}}Q_{1,y-1} + p_{\mathrm{u}}p_{\mathrm{d}}Q_{2,y} + q_{\mathrm{u}}q_{\mathrm{d}}, \qquad y > 1. \tag{23}$$

Adjacent to the wall, corresponding to $y = m$, we sum (9) for $t \geq 1$, adjusting for the $t = 0$ term in the $m = 1$ case as given by (13). The resultant recurrences for $Q_{m,y}$ adjacent to the wall are

$$Q_{m,m} = p_{\mathrm{u}}q_{\mathrm{w}}Q_{m,m+1} + q_{\mathrm{u}}p_{\mathrm{w}}Q_{m,m-1} + p_{\mathrm{u}}p_{\mathrm{w}}Q_{m+1,m} + q_{\mathrm{u}}q_{\mathrm{w}}Q_{m-1,m}, \qquad m > 1, \tag{24}$$

$$Q_{1,1} = p_{\mathrm{u}}q_{\mathrm{w}}Q_{1,2} + q_{\mathrm{u}}p_{\mathrm{w}}Q_{1,0} + p_{\mathrm{u}}p_{\mathrm{w}}Q_{2,1} + q_{\mathrm{u}}q_{\mathrm{w}}. \tag{25}$$

On the wall, for $y = m - 1$, we sum (14) and adjust for $r_0(1, 1)$ as given by (18). The resultant recurrences for $Q_{m,y}$ on the wall are

$$Q_{m,m-1} = p_{\mathrm{u}}Q_{m,m} + q_{\mathrm{u}}Q_{m-1,m-1}, \qquad m > 1, \tag{26}$$

$$Q_{1,0} = p_{\mathrm{u}}Q_{1,1} + q_{\mathrm{u}}. \tag{27}$$

We will now solve the six recurrences in (22)–(27) to find $Q_{m,y}$.

2.2.2. Form of solution. We search for separable solutions of the general recurrence in (22), defining $Q_{m,y}$ to be of the form

$$Q_{m,y} = M(m)Y(y). \tag{28}$$

We substitute this into (22), separating the terms to obtain

$$\frac{M(m) - p_{\mathrm{u}}p_{\mathrm{d}}M(m+1) - q_{\mathrm{u}}q_{\mathrm{d}}M(m-1)}{M(m)} = \frac{p_{\mathrm{u}}q_{\mathrm{d}}Y(y+1) + q_{\mathrm{u}}p_{\mathrm{d}}Y(y-1)}{Y(y)}, \tag{29}$$

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where we have assumed $M(m) \neq 0$, $Y(y) \neq 0$—noting that either of these would lead to $Q_{m,y} = 0$, which does not satisfy the boundary conditions.

We allow both sides of (29) to be equal to a separation variable s, and rearrange to get the recurrences

$$p_{\mathrm{u}}p_{\mathrm{d}}M(m+1) + (s-1)M(m) + q_{\mathrm{u}}q_{\mathrm{d}}M(m-1) = 0, \tag{30}$$

$$p_{\mathrm{u}}q_{\mathrm{d}}Y(y+1) - sY(y) + q_{\mathrm{u}}p_{\mathrm{d}}Y(y-1) = 0. \tag{31}$$

Following the work done on the biased *dry* wall case in [10], we choose $s = p_{\mathrm{u}}q_{\mathrm{d}} + q_{\mathrm{u}}p_{\mathrm{d}}$, and assume an exponential form for $M(m)$ and $Y(y)$. When substituted into (30) and (31), this leads to the solutions

$$M(m) = 1 \quad \text{or} \quad \left(\frac{q_{\mathrm{u}}q_{\mathrm{d}}}{p_{\mathrm{u}}p_{\mathrm{d}}}\right)^m \qquad \text{and} \qquad Y(y) = 1 \quad \text{or} \quad \left(\frac{q_{\mathrm{u}}p_{\mathrm{d}}}{p_{\mathrm{u}}q_{\mathrm{d}}}\right)^y, \tag{32}$$

which are not sufficient alone to satisfy all conditions on $Q_{m,y}$. Guided by the work of [10] and [12] we now choose $s = p_{\mathrm{u}}p_{\mathrm{d}} + q_{\mathrm{u}}q_{\mathrm{d}}$, which leads to further exponential solutions of the form

$$M(m) = \left(\frac{q_{\mathrm{u}}}{p_{\mathrm{u}}}\right)^m \quad \text{or} \quad \left(\frac{q_{\mathrm{d}}}{p_{\mathrm{d}}}\right)^m \qquad \text{and} \qquad Y(y) = \left(\frac{q_{\mathrm{u}}}{p_{\mathrm{u}}}\right)^y \quad \text{or} \quad \left(\frac{p_{\mathrm{d}}}{q_{\mathrm{d}}}\right)^y. \tag{33}$$

We note that all forms of solution found thus far have been identical to those in the dry wall case. This is expected as at this stage we are seeking a form of solution which satisfies the general recurrence away from the wall, which corresponds to directed compact percolation in the bulk.

We now discard the solutions from (32) to (33) which are unbounded as y increases in the high-density region, in addition to the constant term, as we know that the result must tend to the bulk as y increases. That is

$$\lim_{y\to\infty} Q_{m,y} = \left(\frac{q_{\mathrm{u}}q_{\mathrm{d}}}{p_{\mathrm{u}}p_{\mathrm{d}}}\right)^m. \tag{34}$$

Combining the remaining terms from (32) to (33), we have a trial form of solution for the probability of a finite cluster being grown in the case of biased directed compact percolation near a damp wall

$$Q_{m,y} = \begin{cases} A_1\left(\frac{q_{\mathrm{u}}q_{\mathrm{d}}}{p_{\mathrm{u}}p_{\mathrm{d}}}\right)^m + B_1\left(\frac{q_{\mathrm{u}}}{p_{\mathrm{u}}}\right)^{m+y} + C_1\left(\frac{q_{\mathrm{d}}}{p_{\mathrm{d}}}\right)^m\left(\frac{q_{\mathrm{u}}}{p_{\mathrm{u}}}\right)^y, & y \geq m; \\ A_2\left(\frac{q_{\mathrm{u}}q_{\mathrm{d}}}{p_{\mathrm{u}}p_{\mathrm{d}}}\right)^m + B_2\left(\frac{q_{\mathrm{u}}}{p_{\mathrm{u}}}\right)^{2m} + C_2, & y = m-1. \end{cases} \tag{35}$$

We note that this form will satisfy (22), by construction, and now seek to find the coefficients such that it will satisfy all other constraints on $Q_{m,y}$. Substituting (35) into equations (23)–(27) and solving the resultant simultaneous equations, we find the coefficients to be

$$A_1 = 1 \tag{36}$$

$$B_1 = \frac{(1-q_{\mathrm{d}}-q_{\mathrm{u}})(q_{\mathrm{d}}-q_{\mathrm{u}}-(1-q_{\mathrm{u}})q_{\mathrm{w}})}{(q_{\mathrm{d}}-q_{\mathrm{u}})(1-q_{\mathrm{d}}-q_{\mathrm{u}}+q_{\mathrm{u}}q_{\mathrm{w}})} \tag{37}$$

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$$C_1 = -\frac{(1-q_{\rm d}-q_{\rm u})(q_{\rm d}-q_{\rm u}-(1-q_{\rm u})q_{\rm w})}{(q_{\rm d}-q_{\rm u})(1-q_{\rm d}-q_{\rm u}+q_{\rm u}q_{\rm w})} \tag{38}$$

$$A_2 = \frac{q_{\rm w}(1-q_{\rm d})(1-q_{\rm u})(1-2q_{\rm u})}{q_{\rm d}(q_{\rm d}-q_{\rm u})(1-q_{\rm d}-q_{\rm u}+q_{\rm u}q_{\rm w})} \tag{39}$$

$$B_2 = \frac{(1-q_{\rm u})(1-q_{\rm d}-q_{\rm u})(q_{\rm d}-q_{\rm u}-(1-q_{\rm u})q_{\rm w})}{q_{\rm u}(q_{\rm d}-q_{\rm u})(1-q_{\rm d}-q_{\rm u}+q_{\rm u}q_{\rm w})} \tag{40}$$

$$C_2 = 0. \tag{41}$$

So, using these coefficients in (35), we have an expression for the probability of a finite cluster being grown from a given seed.

3. Results

3.1. Result with bias

Using (21), we calculate the percolation probability for a cluster grown, according to biased directed compact percolation, from a seed of width m, with midpoint y units from a damp wall, to be

$$P_{m,y} = 1 - \left(\frac{q_{\rm u}q_{\rm d}}{p_{\rm u}p_{\rm d}}\right)^m - \frac{(p_{\rm u}-q_{\rm d})(p_{\rm u}p_{\rm w}-p_{\rm d})}{(p_{\rm u}-p_{\rm d})(p_{\rm d}-p_{\rm w}q_{\rm u})}\left(\left(\frac{q_{\rm u}}{p_{\rm u}}\right)^m - \left(\frac{q_{\rm d}}{p_{\rm d}}\right)^m\right)\left(\frac{q_{\rm u}}{p_{\rm u}}\right)^y, \qquad y \geq m, \tag{42}$$

$$P_{m,m-1} = 1 - \frac{q_{\rm u}q_{\rm w}(2p_{\rm u}-1)}{(p_{\rm u}-p_{\rm d})(p_{\rm d}-p_{\rm w}q_{\rm u})}\left(\frac{q_{\rm u}q_{\rm d}}{p_{\rm u}p_{\rm d}}\right)^{m-1} - \frac{(p_{\rm u}-q_{\rm d})(p_{\rm u}p_{\rm w}-p_{\rm d})}{(p_{\rm u}-p_{\rm d})(p_{\rm d}-p_{\rm w}q_{\rm u})}\left(\frac{q_{\rm u}}{p_{\rm u}}\right)^{2m-1}, \tag{43}$$

where this result holds for the high-density region, and elsewhere the percolation probability is zero.

3.2. Result without bias

We can consider the unbiased case by setting $p_{\rm u} = p_{\rm d} = p$ in (42) and (43). We rearrange, factoring out apparent singularities, to get an expression for the percolation probability in the unbiased case,

$$P_{m,y}(p,p,p_{\rm w}) = \begin{cases} 1 - \left(\dfrac{q}{p}\right)^{2m} - \dfrac{q_{\rm w}(1-2q)m}{q(1-2q+qq_{\rm w})}\left(\dfrac{q}{p}\right)^{m+y}, & y \geq m, \\ 1 - \left(\dfrac{q}{p}\right)^{2m-1} - \dfrac{q_{\rm w}(1-2q)(m-p)}{q(1-2q+qq_{\rm w})}\left(\dfrac{q}{p}\right)^{2m-1}, & y = m-1, \end{cases} \tag{44}$$

which holds for $p > \frac{1}{2}$, the percolating region for the unbiased case as found in [15]. This result can also be verified by solving the recurrence relations in the unbiased case [19].

4. Analysis

4.1. Percolating region

The percolating region corresponds to the region where $P_{m,y} > 0$. Setting the expressions in (42) equal to zero, we rearrange with the assumption that it must hold for all m, y.

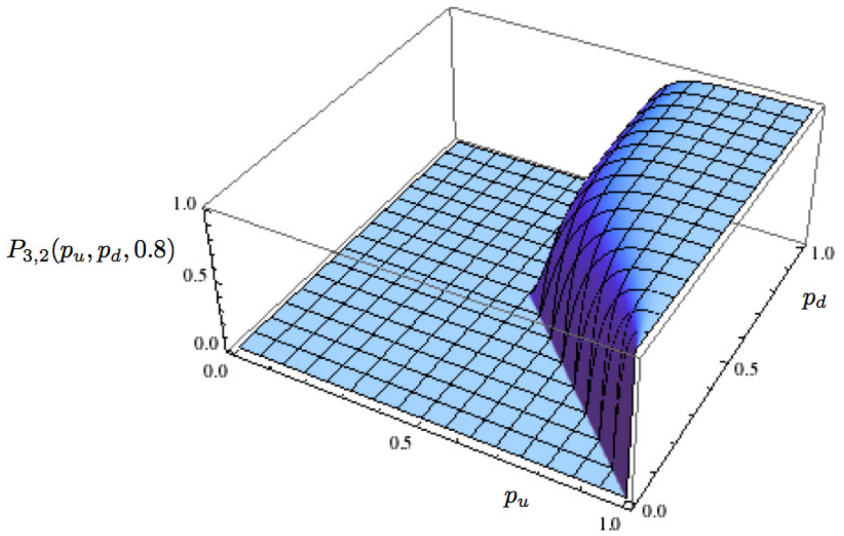

Figure 2. The percolation probability for a cluster beginning on the wall with seed width 3, where the wall occupation probability has been set to $p_\mathrm{w} = 0.8$ so that the dependence on p_u and p_d can be seen.

This means that the transition point should be independent of seed width or distance from the wall. We find that $P_{m,y} = 0$ when either $p_\mathrm{u} = \frac{1}{2}$ or $p_\mathrm{u} + p_\mathrm{d} = 1$, regardless of p_w. The positive $P_{m,y}$ region is to the right of these lines, as shown in figure 4, identical to the high-density region in the dry wall case. We also see very clearly the percolating region in the graph of percolation probability given in figure 2. Similarly, the percolating region in the unbiased case is $p > \frac{1}{2}$, as can be seen in figure 3. This corresponds to the section of the biased percolating region where $p_\mathrm{u} = p_\mathrm{d}$.

Hence in the biased damp case we have a critical curve consisting of two line segments intersecting at the crossover point $p_\mathrm{u} = \frac{1}{2}, p_\mathrm{d} = \frac{1}{2}$. On this critical curve there is a phase transition, between clusters of only finite sizes being grown from a seed and the possibility of an infinite cluster. To analyse the critical behaviour we consider separately the three 'sections' of the critical curve: for $p_\mathrm{d} > \frac{1}{2}$, where the phase transition occurs on the line $p_\mathrm{u} = \frac{1}{2}$; for $p_\mathrm{d} < \frac{1}{2}$, where the phase transition occurs on the line $p_\mathrm{d} + p_\mathrm{u} = 1$; and $p_\mathrm{d} \approx \frac{1}{2}$, where the phase transition occurs at the crossover point.

4.2. Asymptotic form, $p_\mathrm{d} > \frac{1}{2}$

In the region $p_\mathrm{d} > \frac{1}{2}$, the phase transition occurs at the line $p_\mathrm{u} = \frac{1}{2}$. Using Mathematica, we take the series expansion of the expression in (42) in p_u about $\frac{1}{2}$ with the result

$$P_{m,y} \cong f(p_\mathrm{d}, p_\mathrm{w}, m, y)(2p_\mathrm{u} - 1), \qquad y \geq m, \tag{45}$$

$$P_{m,m-1} \cong g(p_\mathrm{d}, p_\mathrm{w}, m)(2p_\mathrm{u} - 1), \tag{46}$$

where

$$f(p_\mathrm{d}, p_\mathrm{w}, m, y) = 2\left(m + m\left(\frac{q_\mathrm{d}}{p_\mathrm{d}}\right)^m + \left(1 - \left(\frac{q_\mathrm{d}}{p_\mathrm{d}}\right)^m\right)\frac{2p_\mathrm{d}q_\mathrm{w}(y-1) + p_\mathrm{w}y - 4p_\mathrm{d}q_\mathrm{d}y}{(2p_\mathrm{d}-1)(2p_\mathrm{d}+q_\mathrm{w}-1)}\right), \tag{47}$$

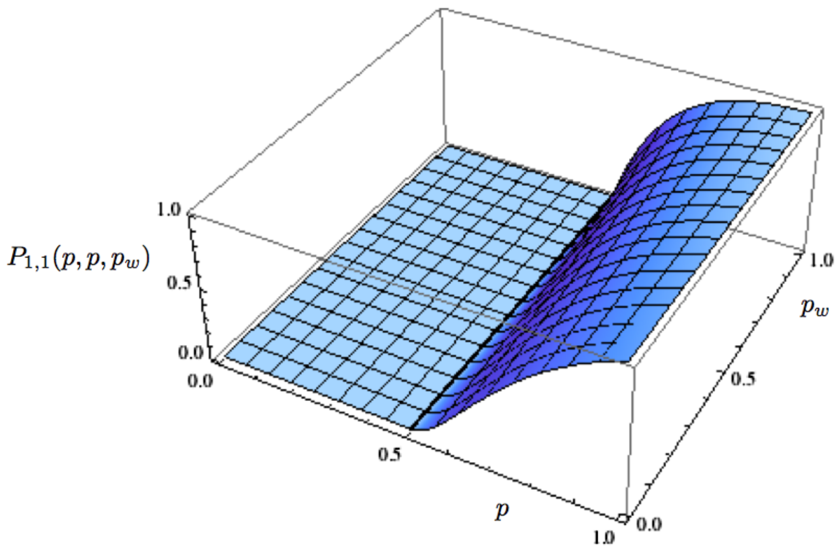

Figure 3. The percolation probability for a cluster beginning with a seed of a single site adjacent to the wall, in the unbiased case of directed compact percolation near a damp wall. The line at $p = \frac{1}{2}$ denotes the transition from the low-density to the high-density region.

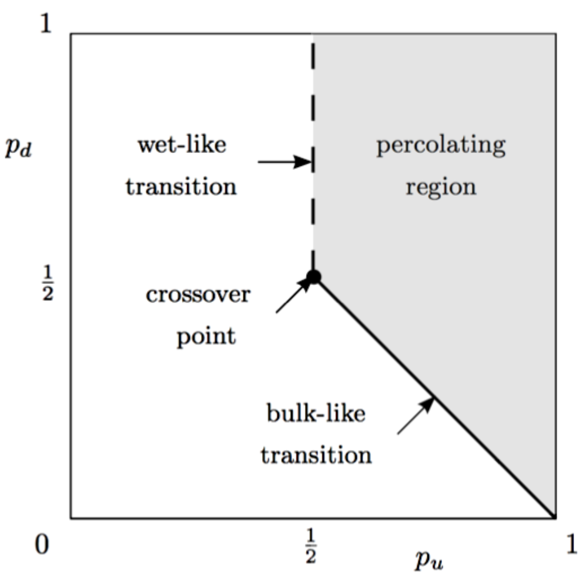

Figure 4. The critical curve for biased directed compact percolation near a damp wall, for any p_{w}. This replicates the results near a dry wall shown in [10].

$$g(p_{\mathrm{d}}, p_{\mathrm{w}}, m) = 4m + \frac{2(1 - 4p_{\mathrm{d}}^3 - 4p_{\mathrm{d}}^2(q_{\mathrm{w}} - 2) - q_{\mathrm{w}} - p_{\mathrm{d}}(5p_{\mathrm{w}} + q_{\mathrm{w}}\,(q_{\mathrm{d}}/p_{\mathrm{d}})^m))}{(1 - 3p_{\mathrm{d}} + 2p_{\mathrm{d}}^2)(2p_{\mathrm{d}} + q_{\mathrm{w}} - 1)}. \tag{48}$$

So we see that the critical exponent

$$\beta^{\mathrm{bias}} = 1, \qquad \text{for } p_{\mathrm{d}} > \tfrac{1}{2}. \tag{49}$$

The physical interpretation of the behaviour in this region is that of a push towards the surface, because the downward probability is greater than half. Thus it is not surprising that the critical exponent in this region is the same as the wet wall critical exponent. We hence call this section of the critical curve the *wet-like transition*, as shown in figure 4.

4.3. Asymptotic form, $p_{\mathrm{d}} < \frac{1}{2}$

In the region $p_{\mathrm{d}} < \frac{1}{2}$, the phase transition occurs at the line $p_{\mathrm{u}}+p_{\mathrm{d}} = 1$. Using Mathematica, we take the series expansion of the expression in (42) in p_{u} about $1 - p_{\mathrm{d}}$ with the result

$$P_{m,y} \cong h(p_{\mathrm{d}}, p_{\mathrm{w}}, m, y)(p_{\mathrm{u}} + p_{\mathrm{d}} - 1), \tag{50}$$

$$P_{m,m-1} \cong j(p_{\mathrm{d}}, p_{\mathrm{w}}, m)(p_{\mathrm{u}} + p_{\mathrm{d}} - 1), \tag{51}$$

where

$$h(p_{\mathrm{d}}, p_{\mathrm{w}}, m, y) = \frac{m}{p_{\mathrm{d}}q_{\mathrm{d}}} - \left(\left(\frac{q_{\mathrm{d}}}{p_{\mathrm{d}}}\right)^{m-y} - \left(\frac{p_{\mathrm{d}}}{q_{\mathrm{d}}}\right)^{m+y}\right) \frac{q_{\mathrm{d}}(1 + p_{\mathrm{w}}) - 1}{q_{\mathrm{w}}p_{\mathrm{d}}(2p_{\mathrm{d}} - 1)}, \tag{52}$$

$$j(p_{\mathrm{d}}, p_{\mathrm{w}}, m) = \frac{m}{p_{\mathrm{d}}q_{\mathrm{d}}} + \frac{1}{p_{\mathrm{d}}q_{\mathrm{w}}} - \left(\frac{p_{\mathrm{d}}}{q_{\mathrm{d}}}\right)^{2m} \left[\frac{q_{\mathrm{d}}}{p_{\mathrm{d}}^2 q_{\mathrm{w}}} + \frac{q_{\mathrm{d}}^2}{p_{\mathrm{d}}^2(2p_{\mathrm{d}} - 1)}\right] + \frac{1 - p_{\mathrm{d}} - p_{\mathrm{d}}^2}{q_{\mathrm{d}}p_{\mathrm{d}}(2p_{\mathrm{d}} - 1)}. \tag{53}$$

So we see that the critical exponent

$$\beta^{\mathrm{bias}} = 1, \qquad \text{for } p_{\mathrm{d}} < \tfrac{1}{2}. \tag{54}$$

The physical interpretation of this region is of a push away from the surface. For $p_{\mathrm{d}} < \frac{1}{2}$, there is decreased growth in the direction towards the wall. Along the line $p_{\mathrm{u}} = 1 - p_{\mathrm{d}}$ we have that as p_{d} decreases, p_{u} increases—meaning that there is a definite preference for growth away from the wall. We hence call this section of the critical curve the *bulk-like transition*, as shown in figure 4.

4.4. Asymptotic form, $p_{\mathrm{d}} \approx \frac{1}{2}$

To find the critical exponent at the crossover point, where $p_{\mathrm{u}} = \frac{1}{2}$ and $p_{\mathrm{d}} = \frac{1}{2}$, we consider approaching the point from a distance r along a line at angle θ to the p_{u} axis. We can hence express the coordinates of the point along these lines in terms of r and θ as

$$p_{\mathrm{u}} = \tfrac{1}{2} + r\cos\theta, \qquad p_{\mathrm{d}} = \tfrac{1}{2} + r\sin\theta. \tag{55}$$

We substitute this into the expression for the percolation probability in (42), and expand about $r = 0$, to find the asymptotic behaviour as $r \to 0$,

$$P_{m,y} \cong \frac{16}{q_{\mathrm{w}}} m(p_{\mathrm{w}} + q_{\mathrm{w}}y)r^2\cos\theta(\cos\theta + \sin\theta), \qquad y \geq m, \tag{56}$$

$$P_{m,m-1} \cong \frac{8}{q_{\mathrm{w}}}(2m^2q_{\mathrm{w}} + (1 - 2q_{\mathrm{w}})(2m - 1))r^2\cos\theta(\cos\theta + \sin\theta). \tag{57}$$

This shows that the exponent

$$\beta^{\mathrm{damp}} = 2 \tag{58}$$

along any such line approaching $p_{\mathrm{u}} = \frac{1}{2}, p_{\mathrm{d}} = \frac{1}{2}$, as was found in the case of a bias near a dry wall in [10]. However, at any other point on the critical curve, only one of the factors

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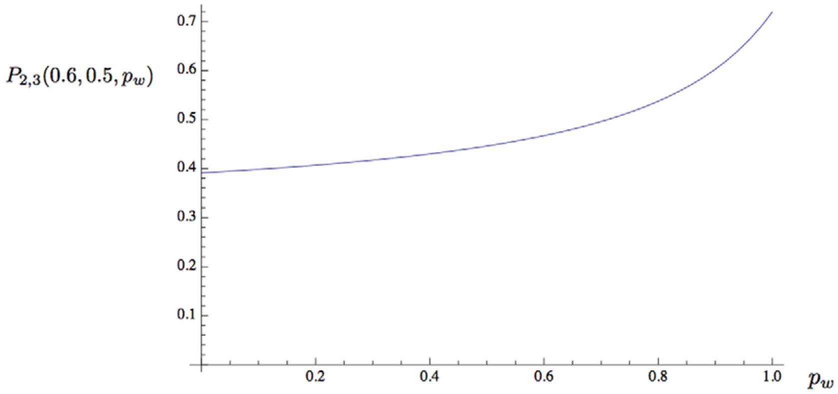

Figure 5. The percolation probability for varying p_w, with all other variables fixed.

$(2p_u - 1)$ or $(p_u + p_d - 1)$ is equal to zero. This means that everywhere except at the crossover point we have $\beta = 1$, as found in the $p_d > \frac{1}{2}$ and $p_d < \frac{1}{2}$ regions, which is the critical exponent in the bulk and wet wall cases.

4.5. Effect of varying each of the variables

We can consider the effect of varying each of the variables present in our general solution for the percolation probability in (42) and (43), that is each of p_u, p_d, p_w, m and y. The effect of varying p_u or p_d can be seen in figure 2, and it is these variables which determine the percolating region as shown in figure 4. The critical behaviour also changes depending on the relative relationship of p_u and p_d. Any bias away from the wall tends to the bulk case, and any bias towards the wall tends to the wet wall case; these results mimic the findings of biased growth near a dry wall [10]. The critical exponent, when a bias is present, is equal to the bulk/wet value of 1; however, in the unbiased damp case the critical exponent is equal to 2. So the unbiased case of directed compact percolation near a damp wall is a special case, as any bias either towards or away from the wall leads us back to the bulk critical exponent.

Varying the wall occupation probability p_w also affects the percolation probability, as seen in figure 5, although it does not change the percolating region. In the unbiased case $p_u = p_d$, the critical exponent differs depending on whether $p_w < 1$ (damp/dry exponent $\beta = 2$) or $p_w = 1$ (wet/bulk exponent $\beta = 1$). Increasing the seed width, m, for clusters a fixed distance from the wall, leads to a strong increase in the percolation probability, as shown in figure 6. This follows naturally from the rules of directed compact percolation, since a cluster with a large seed width is much less likely to terminate than one with a small seed width. The distance from the wall, measured by y, also affects the percolation probability, although the effect becomes more subtle as the distance from the wall increases. A cluster beginning on the wall or adjacent to the wall is naturally more constrained in its growth than a cluster which begins away from the wall, since the sites below the wall are unable to be occupied. As a result, clusters beginning nearer the wall are less likely to grow into infinite clusters in the high-density region, and we note accordingly in figure 7 that the percolation probability in the high-density region is lowest for clusters beginning on the wall, and increases as we move away from the wall.

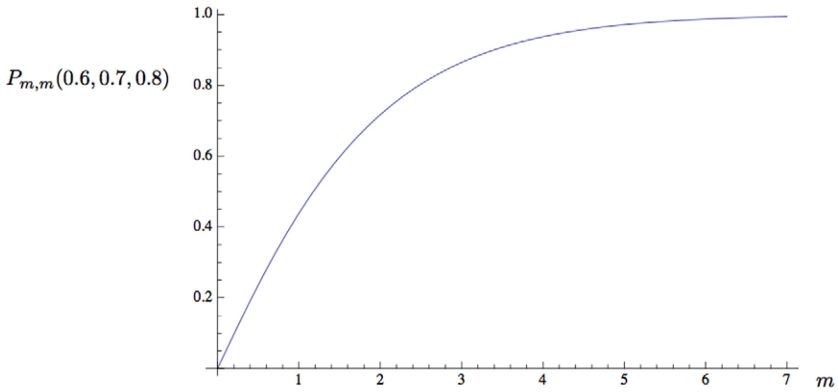

Figure 6. The percolation probability for varying m, for clusters beginning adjacent to the wall ($y = m$), with all other variables fixed.

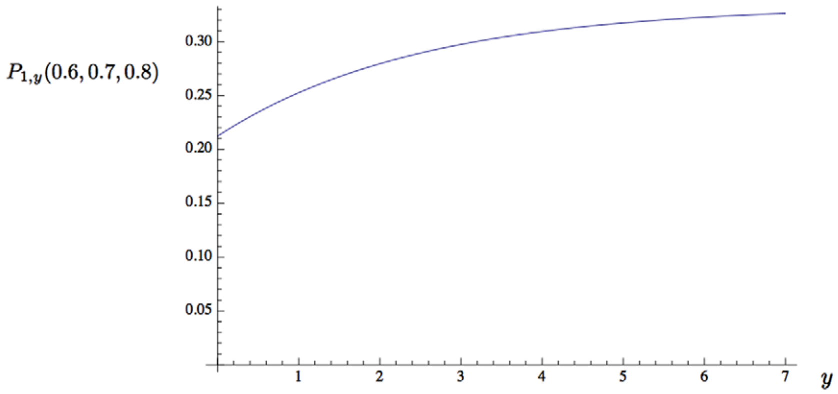

Figure 7. The percolation probability for varying y, with all other variables fixed.

5. Conclusions and special cases

We have found an exact general expression for the percolation probability for directed compact percolation near a damp wall, with general seed width at any distance from the wall and allowing for bias. This formulation allows all previously studied cases to be obtained as special cases.

5.1. Special cases

5.1.1. Bulk comparison. In the case of a cluster beginning away from the wall, we can take the limit $y \to \infty$ to find an expression for the percolation probability in the bulk limit,

$$\lim_{y\to\infty} P_{m,y} = 1 - \left(\frac{q_\mathrm{u} q_\mathrm{d}}{p_\mathrm{u} p_\mathrm{d}}\right)^m, \tag{59}$$

which reproduces the expression for the biased bulk case found in [9].

5.1.2. Wet wall comparison We can use our general expression to derive the wet wall result also, by setting $p_\mathrm{w} = p_\mathrm{d} = 1$. In this way our expression in (42) reduces nicely to

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the wet wall expression,

$$P_{m,y}|_{p_{\rm w}=p_{\rm d}=1} = 1 - \left(\frac{q_{\rm u}}{p_{\rm u}}\right)^{2m-1}, \tag{60}$$

which reproduces the expression for the wet wall case found in [10].

5.1.3. Dry wall comparison. The result for directed compact percolation near a dry wall, using biased growth, can be obtained from (42) simply by setting $p_{\rm w} = p_{\rm d}$. This replaces the wall with a row of sites which are occupied based on the bulk downward probability, and the row below becomes the effective dry wall. Setting $p_{\rm w} = p_{\rm d}$ in (42), we have

$$P_{m,y}(p_{\rm u}, p_{\rm d}, p_{\rm d}) = 1 - \left(\frac{q_{\rm u}q_{\rm d}}{p_{\rm u}p_{\rm d}}\right)^{m} - \frac{p_{\rm u} - q_{\rm d}}{q_{\rm u} - q_{\rm d}}\left[\left(\frac{q_{\rm u}}{p_{\rm u}}\right)^{m} - \left(\frac{q_{\rm d}}{p_{\rm d}}\right)^{m}\right]\left(\frac{q_{\rm u}}{p_{\rm u}}\right)^{y+1}, \tag{61}$$

which agrees with the expression for the biased dry wall case found in [10]. We can similarly apply the expression in (44) to the unbiased dry wall case, by setting $p_{\rm w} = p$, with the result

$$P_{m,y}(p, p, p) = 1 - \left(\frac{q}{p}\right)^{2m} - \frac{m(2p-1)}{p^2}\left(\frac{q}{p}\right)^{m+y}, \tag{62}$$

which rederives the percolation probability in the unbiased dry wall case found in [10].

5.1.4. Damp wall comparison. The result in (44) generalizes the result of the specific unbiased case considered in [15], of a seed of width one beginning adjacent to a damp wall, and we can see

$$P_{1,1}(p, p, p_{\rm w}) = \begin{cases} \dfrac{(1-2q)^2}{(1-q)^2(1-2q+qq_{\rm w})}, & p > \dfrac{1}{2}; \\ 0, & p \le \dfrac{1}{2}. \end{cases} \tag{63}$$

which is the same as the result found in [15] using a mapping to pairs of weighted directed walks.

So the expression for percolation probability in (42) has as special cases all previously studied cases of directed compact percolation on a square lattice: that is, in the bulk, near a wet wall, near a dry wall and near a damp wall, in both the biased and unbiased cases.

5.2. Conclusion

We have analysed our full expression pointing out behaviour in terms of each variable and providing scaling analysis near the percolation transition which depends on the variable, including the special crossover point where the system changes from bulk-like behaviour to 'wet-wall'-like behaviour. The exponents in these two cases are the same and it is the crossover point where the system is unbiased that is special. We conclude that whether the wall is dry or damp does not effect the gross behaviour of the percolating system.

Acknowledgments

Financial support from the Australian Research Council via its support for the Centre of Excellence for Mathematics and Statistics of Complex Systems and its Discovery Program is gratefully acknowledged by the authors.

References

[1] Kesten H, 1982 *Percolation Theory for Mathematicians* (Boston, MA: Birkhäuser)
[2] Stauffer D and Aharony A, 1991 *Introduction to Percolation Theory* 2nd edn (London: Taylor and Francis)
[3] Smirnov S and Werner W, *Critical exponents for two-dimensional percolation*, 2001 *Math. Res. Lett.* **8** 729 (arXiv:math/0109120v2)
[4] Cardy J L and Grassberger P, *Epidemic models and percolation*, 1985 *J. Phys. A: Math. Gen.* **18** L267
[5] Hinrichsen H, Jiménez-Dalmaroni A, Rozov Y and Domany E, *Flowing sand: a physical realization of directed percolation*, 1999 *Phys. Rev. Lett.* **83** 4999
[6] Schulman L S and Seiden P E, *Percolation analysis of stochastic models of galactic evolution*, 1982 *J. Stat. Phys.* **27** 83
[7] Stanley H E, Andrade J S Jr, Havlin S, Makse H A and Suki B, *Percolation phenomena: a broad-brush introduction with some recent applications to porous media, liquid water, and city growth*, 1999 *Physica* A **266** 5
[8] Domany E and Kinzel W, *Equivalence of cellular automata to Ising models and directed percolation*, 1984 *Phys. Rev. Lett.* **53** 311
[9] Essam J W, *Directed compact percolation: cluster size and hyperscaling*, 1989 *J. Phys. A: Math. Gen.* **22** 4927
[10] Essam J W and TanlaKishani D, *Directed compact percolation near a wall: I. Biased growth*, 1994 *J. Phys. A: Math. Gen.* **27** 3743
[11] Bidaux R and Privman V, *Surface-to-bulk crossover in directed compact percolation*, 1991 *J. Phys. A: Math. Gen.* **24** L839
[12] Lin J C, *Exact results for directed compact percolation near a nonconducting wall*, 1992 *Phys. Rev.* A **45** R3394
[13] Essam J W and Guttmann A J, *Directed compact percolation near a wall: II. Cluster length and size*, 1995 *J. Phys. A: Math. Gen.* **28** 3591
[14] Brak R and Essam J W, *Directed compact percolation near a wall: III. Exact results for the mean length and number of contacts*, 1999 *J. Phys. A: Math. Gen.* **32** 355
[15] Lonsdale H, Brak R, Essam J W, Owczarek A L and Rechnitzer A, *On directed compact percolation near a damp wall*, 2009 *J. Phys. A: Math. Theor.* **42** 125001
[16] Essam J W, Lonsdale H and Owczarek A L, *Mean length of finite clusters in directed compact percolation near a damp wall*, 2011 *J. Stat. Phys.* **145** 639
[17] Lonsdale H, Essam J W and Owczarek A L, *Directed compact percolation near a damp wall: mean length and mean number of wall contacts*, 2011 *J. Phys. A: Math. Theor.* **44** 505003
[18] Lonsdale H, Jensen I, Essam J W, Owczarek A L and Rechnitzer A, *Analysis of mean cluster size in directed compact percolation near a damp wall*, in preparation
[19] Essam J W, 2011 private communication