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Exact results for a directed polymer model related to quantum entanglement in far from equilibrium stationary states

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Abstract. A directed polymer model has been used by Alcaraz *et al* to reflect properties related to models of quantum entanglement in far from equilibrium stationary states. Here we calculate exactly one property related to entanglement: the average height of the polymer from the surface. In doing so we extend a well known method of exact solution for directed polymer models. Of interest from the polymer viewpoint is that the average height is asymptotically independent of the stickiness in the desorbed phase.

Keywords: solvable lattice models, classical phase transitions (theory), critical exponents and amplitudes (theory)

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1. Introduction and model

In the field of quantum information the property of entanglement plays a key role (see [2] and references therein). It is also of great interest to those studying quantum critical phenomena [3]–[12] since the scaling behaviour of the entanglement is a sign of criticality. The entanglement is a property of one part of a quantum system that is divided into parts. Consider a quantum system $\mathcal{C}$ in a pure state which can be considered as a sum of two parts $\mathcal{A}$ and $\mathcal{B}$. The von Neumann entanglement entropy S_q is then defined as

$$S_q(\mathcal{A}) = -\mathrm{Tr}(\rho_\mathcal{A} \ln \rho_\mathcal{A}) \tag{1.1}$$

where $\rho_\mathcal{A} = \mathrm{Tr}_\mathcal{B}\rho$ with ρ being the density matrix of the total system, and with $S_q(\mathcal{A}) = S_q(\mathcal{B})$. Let us consider one-dimensional quantum spin chains of total size n. If the system is non-critical the entanglement entropy of the ground state will saturate to a constant. However, if the ground state of an infinite system is critical then the entropy diverges with the size $n_\mathcal{A}$ of the subsystem $\mathcal{A}$ as

$$S_q(n_\mathcal{A}) \sim \gamma \ln n_\mathcal{A} + C \qquad \text{for } n \gg n_\mathcal{A} \tag{1.2}$$

where γ is proportional to the central charge c of the corresponding conformal field theory [5] and depends on the kinds of boundary conditions that have been imposed. Here C is a non-universal constant. Furthermore, the finite-size scaling behaviour is predicted to be

$$S_q(n_\mathcal{A}, n) \approx \gamma \ln[n \sin(\pi n_\mathcal{A}/n)/\pi] + C. \tag{1.3}$$

Note that averaging over $n_\mathcal{A}$ implies a logarithmic divergence in n.

Alcaraz *et al* [1] have recently considered the shared information in ground states which are superpositions of valence bond states. They considered ground states where the coefficients are all real non-negative and so could be considered as probabilities of configurations. Hence they considered the shared information in a bipartition of a classical rather than a quantum system. However, the ground states that they studied can describe the equilibrium problems of spin 1/2 $SU(2)$ symmetric one-dimensional quantum chains [4] as well as the probability distribution functions (PDFs) of stationary states of

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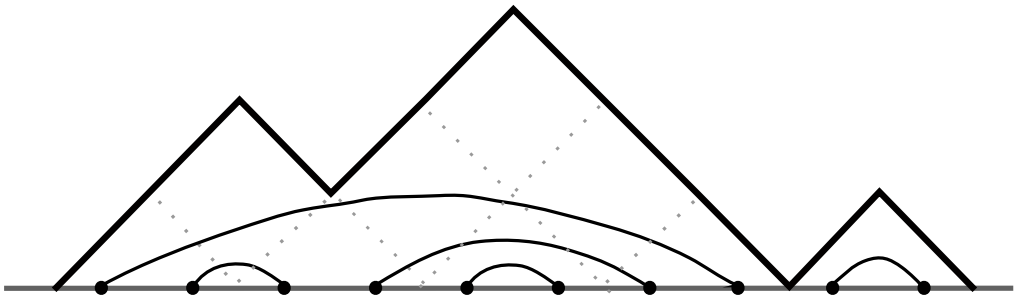

Figure 1. An example of a link pattern and the corresponding Dyck path of length $n = 10$.

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stochastic processes. In particular, they presented four estimators of shared information and compared them in two models. In both models the configuration spaces were Dyck paths. These paths were derived from the consideration of an open one-dimensional system of n sites (n even) where the sites are connected by $n/2$ nonintersecting links; see figure 1. The links were seen as $U_q(sl(2))$ singlets. Importantly, there is a bijection between the link patterns and Dyck paths.

The differences between the two models are the weightings associated with different path configurations. One of the two models considered [1] was a directed polymer model where the Dyck paths are seen as conformations of a polymer and visits of the polymer to the surface are weighted. This means that the model is a well known model of polymer adsorption [13, 14]. Also, one of the four estimators of shared information was the *valence bond entanglement entropy* [15, 16, 6] which is the average height of the Dyck path at a site $n_{\mathcal{A}}$ for a system of size n. Despite the fact that the adsorption model has been well studied, the average height for a large system has not been calculated previously as a function of the weighting parameter, even when averaged over $n_{\mathcal{A}}$. One reason for this is that naively it requires the solution of a more general model where the area under the polymer is weighted. While this can be done, one is then required to use q-series asymptotics to provide the required results. Here we calculate the needed quantities without recourse to q-series and the solution of the generalized model, by extending the so-called Temperley method [17, 18]. Usually in the solution of directed polymer models by the Temperley method [17, 18] a second-order difference equation is obtained; here we derive a set of two coupled equations that can be solved for the quantity of interest.

The model. The model that we focus on is that of adsorbing Dyck paths at a wall [13, 14]. We define a more general model where the sum of heights of the walk is also weighted, though we shall set this weight to 1 in our calculations.

Dyck paths are directed walks on $\mathbb{Z}^2$ starting at $(0, 0)$ and ending on the line $y = 0$ at $(n, 0)$, which have no vertices with negative y-coordinates, and which have steps in the $(1, 1)$ and $(1, -1)$ directions. We consider there to be a wall at $y = 0$. Let us define coordinates of the sites of the walk to be (i, r_i) for $i = 0, \ldots, n$; we then have $r_0 \equiv 0$ and $r_n = 0$ by construction for Dyck paths. We add an energy for sites of the walk that lie on the surface (wall) at $y = 0$ other than at the origin to give the adsorbing polymer model; see figure 2. An energy $-J$ is added for each such *visit*. We define a Boltzmann weight

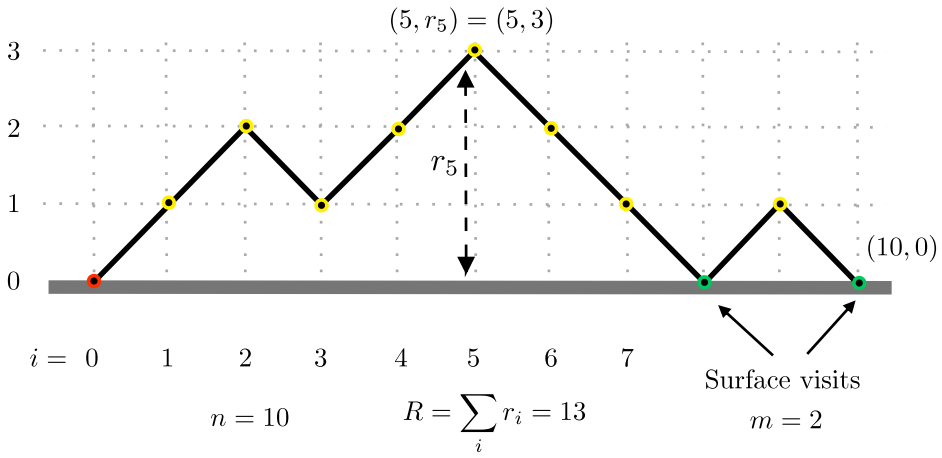

Figure 2. An example of a Dyck path of length $n = 10$ given by the values of the defining variables r_i, $i = 0, 1, \ldots, 10$. These give the heights of the sites above the surface; here we have $r_0 = 0, r_1 = 1, r_2 = 2, r_3 = 1, r_4 = 2, r_5 = 3, r_6 = 2, r_7 = 1, r_8 = 0, r_9 = 1, r_{10} = 0$. The number of visits to the surface, not counting the origin, is $m = 2$, so the Boltzmann weight of this configuration is κ^2. The total height $R = \sum_{i=0}^{10} r_i = 13$ for this configuration.

$\kappa = \mathrm{e}^{\beta J}$ associated with these visits, where $\beta = 1/k_{\mathrm{B}}T$, k_{B} is Boltzmann's constant and T is the absolute temperature. The partition function of our model is

$$\hat{Z}_n(\kappa) = \sum_{\psi_n} \kappa^{m(\psi_n)} \tag{1.4}$$

where ψ_n are Dyck path configurations and $m(\psi_n)$ are the number of visits to the wall by the Dyck path.

The quantity on which we focus is the average height of the sites of the walk above the wall, $\mathcal{R}(n, \kappa)$, that is

$$\mathcal{R}(n, \kappa) \equiv \left\langle \frac{\sum_{i=0}^{n} r_i}{n} \right\rangle = \frac{1}{n} \frac{\sum_{\psi_n} [\sum_{i=0}^{n} r_i(\psi_n)] \kappa^{m(\psi_n)}}{\hat{Z}_n(\kappa)}, \tag{1.5}$$

again where the ψ_n are Dyck path configurations and the $m(\psi_n)$ are the numbers of visits to the wall by the Dyck path. Note that the average height multiplied by the length is the average area between the path and the surface.

In finding our quantities of interest we will need to consider directed walks that end at an arbitrary height r above the wall rather than on the wall. We consider walks with at least one step. Moreover we need to generalize our model to include a weighting for the sum of the heights of the sites of the walk above the surface; this allows us to find the average height. Hence we weight each configuration φ_n^r of length n ending at height r with a weight q^R where

$$R = \sum_{i=1}^{n} r_i. \tag{1.6}$$

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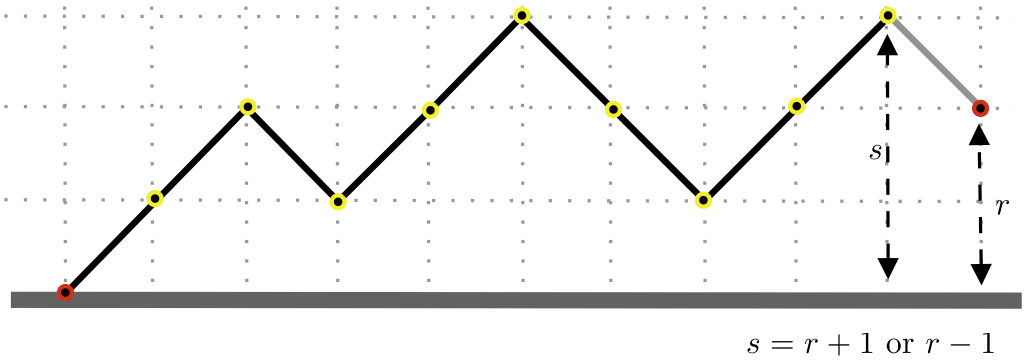

Figure 3. A schematic diagram of the process of adding one column of height r to a configuration ending at height s. If $r \geq 1$ the s can be $r \pm 1$. If $r = 0$ then $s = 1$ only.

Note that the Dyck paths ψ_n are the same elements as φ_n^0.

The partition function for directed walks ending at height r above the wall of length n weighted as described is

$$Z_n^{(r)}(\kappa, q) = \sum_{\varphi_n^r} \kappa^{m(\varphi_n^r)} q^{R(\varphi_n^r)}, \tag{1.7}$$

where $m(\varphi_n^r)$ is the number of visits to the wall by the polymer. Hence

$$\hat{Z}_n(\kappa) = Z_n^{(0)}(\kappa, 1). \tag{1.8}$$

Let us define

$$S_n^{(r)}(\kappa, q) = \sum_{\varphi_n^r} R(\varphi_n^r) \kappa^{m(\varphi_n^r)} q^{R(\varphi_n^r)}. \tag{1.9}$$

We note immediately that

$$S_n^{(r)}(\kappa, q) = q \frac{\mathrm{d} Z_n^{(r)}(\kappa, q)}{\mathrm{d}q}. \tag{1.10}$$

Hence we have

$$\mathcal{R}(n, \kappa) = \frac{1}{n} \frac{S_n^{(0)}(\kappa, 1)}{Z_n^{(0)}(\kappa, 1)}. \tag{1.11}$$

So we *could* calculate $\mathcal{R}(n, \kappa)$ by finding $Z_n^{(0)}(\kappa, q)$ and differentiating, finally setting $q = 1$ after differentiating.

We approach the problem in a slightly different manner. We define generating functions

$$G_r(z, \kappa, q) = \sum_{n=1}^{\infty} Z_n^{(r)}(\kappa, q) z^n \tag{1.12}$$

and

$$H_r(z, \kappa, q) = \sum_{n=1}^{\infty} S_n^{(r)}(\kappa, q) z^n, \tag{1.13}$$

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so we have

$$H_r(z,\kappa,q) = q\frac{\mathrm{d}G_r(z,\kappa,q)}{\mathrm{d}q}, \tag{1.14}$$

and set

$$g_r = G_r(z,\kappa,1) \tag{1.15}$$

and

$$h_r = H_r(z,\kappa,1). \tag{1.16}$$

In this way we have

$$Z_n^{(0)}(\kappa,1) = [z^n]g_0 \tag{1.17}$$

and

$$S_n^{(0)}(\kappa,1) = [z^n]h_0. \tag{1.18}$$

2. Generating functions

We now introduce the method of Temperley [17] which finds a recursion for G_r by the consideration of adding a step (or column) onto an existing walk; see figure 3

By such a consideration we find the recurrence

$$G_r = q^r z(G_{r-1} + G_{r+1}) \tag{2.1}$$

for $r \geq 2$, while

$$G_1 = qz + qz(G_0 + G_2) \tag{2.2}$$

and

$$G_0 = \kappa z G_1. \tag{2.3}$$

The usual Temperley method then continues by solving these recurrences and boundary conditions. We point out that these are second-order recurrences that have power function dependent coefficients; this leads to a q-series solution.

Rather than that, we apply the operator $q(\mathrm{d}/\mathrm{d}q)$ to the recurrences term by term to give

$$H_r = rq^r z(G_{r-1} + G_{r+1}) + q^r z(H_{r-1} + H_{r+1}) \tag{2.4}$$

for $r \geq 2$, while

$$H_1 = qz + qz(G_0 + G_2) + qz(H_0 + H_2) \tag{2.5}$$

and

$$H_0 = \kappa z H_1. \tag{2.6}$$

By setting $q = 1$ in the above recurrences we obtain the two *coupled* systems: firstly,

$$g_r = z(g_{r-1} + g_{r+1}) \tag{2.7}$$

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for $r \geq 2$, while

$$g_1 = qz + qz(g_0 + g_2) \tag{2.8}$$

and

$$g_0 = \kappa z g_1; \tag{2.9}$$

and secondly

$$\begin{aligned} h_r &= rz(g_{r-1} + g_{r+1}) + z(h_{r-1} + h_{r+1}) \\ h_r &= rg_r + z(h_{r-1} + h_{r+1}) \end{aligned} \tag{2.10}$$

for $r \geq 2$, while

$$h_1 = qz + qz(g_0 + g_2) + z(h_0 + h_2) = g_1 + z(h_0 + h_2) \tag{2.11}$$

and

$$h_0 = \kappa z h_1. \tag{2.12}$$

We now have a set of *coupled* difference equations—although importantly they are *constant* coefficient difference equations. Hence they should be solvable via a generalized exponential ansatz, which indeed they are.

We first solve the g_r system: equations (2.7)–(2.9). We assume the ansatz

$$\begin{aligned} g_r &= A(z,\kappa)\lambda(z)^r \qquad \text{for } r \geq 1 \\ g_0 &= B(z,\kappa). \end{aligned} \tag{2.13}$$

Substituting our ansatz into equation (2.7) implies

$$\lambda^2 - z^{-1}\lambda + 1 = 0. \tag{2.14}$$

or rather $\lambda + \lambda^{-1} = z^{-1}$. While there are two solutions to this quadratic, only one corresponds to a formal power series and we have

$$\lambda(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z}. \tag{2.15}$$

On substituting into equation (2.8) one finds

$$A(z,\kappa) = \frac{z}{\lambda(z)[1 - \kappa z - z\lambda(z)]}. \tag{2.16}$$

Finally on substituting into equation (2.9) one completes the solution for g_r by finding

$$g_0 = B = \kappa z A(z,\kappa)\lambda(z) \tag{2.17}$$

and so the generating function for adsorbing Dyck path polymers is the well known result of

$$1 + G_0(z,\kappa,1) = \frac{1 + \sqrt{1 - 4z^2}}{1 - 2\kappa z^2 + \sqrt{1 - 4z^2}} = \frac{2}{2 - \kappa + \kappa\sqrt{1 - 4z^2}}. \tag{2.18}$$

The first few terms of the series expansion are

$$\begin{aligned} 1 + G_0(z,\kappa,1) = 1 &+ \kappa z^2 + (\kappa + \kappa^2)z^4 + (2\kappa + 2\kappa^2 + \kappa^3)z^6 + (5\kappa + 5\kappa^2 + 3\kappa^3 + \kappa^4)z^8 \\ &+ \mathrm{O}(z^{10}) \end{aligned} \tag{2.19}$$

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and at $\kappa = 1$ we have

$$1 + G_0(z, 1, 1) = 1 + z^2 + 2z^4 + 5z^6 + 14z^8 + 42z^{10} + 132z^{12} + \mathrm{O}(z^{14}). \tag{2.20}$$

This solution is well known and can be found in several ways.

Now we turn our attention to the coupled h_r system of equations (2.10)–(2.12). We first combine equations (2.10) and (2.11) as

$$h_{r+1} - z^{-1}h_r + h_{r-1} = -rz^{-1}g_r \tag{2.21}$$

for $r \geq 1$. As we have already solved for g_r, for $A\lambda^r$, we treat this as an inhomogeneous difference equation with known right-hand side. Considering the structure of equation (2.21) we try the ansatz

$$h_r = (ar^2 + br + c)\lambda^r \tag{2.22}$$

with the three undetermined coefficients a, b and c but the same λ as is characterized by equation (2.15). On substitution of equation (2.22) into (2.21) we find

$$a = \frac{-A}{2z(\lambda - \lambda^{-1})} = \frac{A}{2\sqrt{1-4z^2}} \tag{2.23}$$

and

$$b = \frac{A(\lambda + \lambda^{-1})}{2z(\lambda - \lambda^{-1})^2} = \frac{A}{2(1-4z^2)}, \tag{2.24}$$

while c remains arbitrary. We use equation (2.12) to fix c as

$$c = \frac{\kappa z(a+b)\lambda}{1 - \kappa z\lambda}. \tag{2.25}$$

This gives us

$$h_0 = c = \frac{\kappa z A\lambda}{1 - \kappa z\lambda}\left[\frac{1}{2\sqrt{1-4z^2}} + \frac{1}{2(1-4z^2)}\right] \tag{2.26}$$

and so

$$H_0(z, \kappa, 1) = \frac{\kappa z^2}{2(1-\kappa z\lambda)(1-\kappa z - z\lambda)}\left[\frac{1}{\sqrt{1-4z^2}} + \frac{1}{1-4z^2}\right], \tag{2.27}$$

or rather

$$H_0(z, \kappa, 1) = \frac{\kappa z^3}{2(1-\kappa z - z\lambda)^2\lambda}\left[\frac{1}{\sqrt{1-4z^2}} + \frac{1}{1-4z^2}\right]. \tag{2.28}$$

Finally one can write

$$H_0(z, \kappa, 1) = \frac{\kappa z^4}{(1 - 2\kappa z^2 + \sqrt{1-4z^2})^2(1-\sqrt{1-4z^2})}\left[\frac{1}{\sqrt{1-4z^2}} + \frac{1}{1-4z^2}\right]. \tag{2.29}$$

The first few terms of the series expansion are

$$H_0(z, \kappa, 1) = \kappa z^2 + (4\kappa + 2\kappa^2)z^4 + (16\kappa + 10\kappa^2 + 3\kappa^3)z^6 + (64\kappa + 44\kappa^2 + 18\kappa^3 + 4\kappa^4)z^8 + \mathrm{O}(z^{10}) \tag{2.30}$$

and at $\kappa = 1$ we have

$$H_0(z, 1, 1) = z^2 + 6z^4 + 29z^6 + 130z^8 + 562z^{10} + 2380z^{12} + \mathrm{O}(z^{14}). \tag{2.31}$$

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3. Singularity structure

The singularity structure of the generating function $G_0(z, \kappa, 1)$ as a function of z determines the free energy. The reduced free energy is defined as

$$f(\kappa) = -\lim_{n\to\infty} \frac{1}{n} \log(Z_n^{(0)}(\kappa, 1)) \tag{3.1}$$

and is given by

$$f(\kappa) = \log z_s(\kappa), \tag{3.2}$$

where $z_s(\kappa)$ is the closest singularity (on the positive real axis) of the generating function $G_0(z, \kappa, 1)$ in the variable z to the origin.

The key thermodynamic quantity, $\mathcal{M}$, describing the transition is the average number of sites of the walk located in the surface per step of the walk

$$\mathcal{M}(\kappa) = \lim_{n\to\infty} \left\langle \frac{m}{n} \right\rangle = \lim_{n\to\infty} \frac{\sum_{\psi_n} m(\psi_n)\kappa^{m(\psi_n)}}{n\sum_{\psi_n} \kappa^{m(\psi_n)}} = \lim_{n\to\infty} \frac{\kappa}{n} \frac{\mathrm{d}\log(Z_n^{(0)}(\kappa, 1))}{\mathrm{d}\kappa}, \tag{3.3}$$

which implies

$$\mathcal{M}(\kappa) = -\kappa \frac{\mathrm{d}\log z_s(\kappa)}{\mathrm{d}\kappa}. \tag{3.4}$$

That is, the variation of z_s with κ is directly related to the average occupation of the surface by the walk.

The singularity structure of G_0 has been studied and is relatively simple. There clearly is a singularity when the argument of the square root terms is zero at $1 - 4z^2 = 0$, that is, at

$$z = z_d = 1/2. \tag{3.5}$$

For $0 < \kappa < 2$ this is the closest singularity to the origin, so

$$z_s(\kappa) = 1/2 \tag{3.6}$$

for $0 < \kappa < 2$. There is a simple pole in G_0 when the denominator factor $1 - 2\kappa z^2 + \sqrt{1 - 4z^2}$ vanishes; that is, when

$$\kappa = \frac{1 + \sqrt{1 - 4z^2}}{2z^2}. \tag{3.7}$$

Note that any value of z contained in the interval $[0, 1/2]$ gives values of κ in $[2, \infty)$. Hence this can be solved for z giving the value

$$z = z_a = \frac{\sqrt{\kappa - 1}}{\kappa} \leq 1/2 \tag{3.8}$$

valid for $\kappa \geq 2$. Hence

$$z_s(\kappa) = \begin{cases} \frac{1}{2} & \text{for } \kappa \leq 2 \\ \dfrac{\sqrt{\kappa - 1}}{\kappa} & \text{for } \kappa \geq 2 \end{cases} \tag{3.9}$$

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and

$$\mathcal{M}(\kappa) = \begin{cases} 0 & \text{for } \kappa \leq 2 \\ \dfrac{\kappa - 2}{2(\kappa - 1)} & \text{for } \kappa \geq 2. \end{cases} \tag{3.10}$$

This reflects the existence of an adsorption phase transition [19] which has been well characterized [13, 14]. This adsorption transition can be described as follows: for high temperatures (small κ) the average number of sites m of the walk in the surface is bounded ($\mathcal{M} = 0$) while at low temperatures (large κ) the average number of sites of the walk in the surface is proportional to the length n of the walk ($\mathcal{M} > 0$).

For completeness and comparison, and noting they have not appeared explicitly in the literature, we calculate

$$\hat{m}(n, \kappa) = \left\langle \frac{m}{n} \right\rangle \tag{3.11}$$

via

$$V_n(\kappa) = \sum_{\psi_n} m(\psi_n) \kappa^{m(\psi_n)} = n \hat{m}(n, \kappa) Z_n^{(0)}(\kappa, 1). \tag{3.12}$$

We use the generating function

$$M(z, \kappa) = \sum_n V_n(\kappa) z^n = \sum_n \left(\sum_{\psi_n} m(\psi_n) \kappa^{m(\psi_n)} \right) z^n = \kappa \frac{\mathrm{d} G_0(z, \kappa, 1)}{\mathrm{d}\kappa} \tag{3.13}$$

to evaluate V_n. Hence we have

$$M(z, \kappa) = \frac{2\kappa z (1 + \sqrt{1 - 4z^2})}{(1 - 2\kappa z^2 + \sqrt{1 - 4z^2})^2}. \tag{3.14}$$

The first few terms of the series expansion are

$$M(z, \kappa) = \kappa z^2 + (\kappa + 2\kappa^2) z^4 + (2\kappa + 4\kappa^2 + 3\kappa^3) z^6 + (5\kappa + 10\kappa^2 + 9\kappa^3 + 4\kappa^4) z^8 + \mathrm{O}(z^{10}) \tag{3.15}$$

and at $\kappa = 1$ we have

$$M(z, 1) = z^2 + 3z^4 + 9z^6 + 28z^8 + 90z^{10} + 297z^{12} + \mathrm{O}(z^{14}). \tag{3.16}$$

4. Asymptotic evaluations

The generating function H_0 clearly has the same singularities as both G_0 and M since the same denominator factors exist in each and the same algebraic terms exist in each. Let us first note that each of the generating functions for Dyck paths is a power series in z^2 since all Dyck paths are of even length. Singularities at $z = z_s > 0$ are duplicated at $z = -z_s$, with similar asymptotics in z near those points. For $\kappa < 2$ one can expand around the singularity at $z = z_d = 1/2$ to find

$$G_0(z, \kappa, 1) = \frac{\kappa}{2 - \kappa} - \frac{4\kappa}{(2 - \kappa)^2} \sqrt{z_d - z} + \mathrm{O}((z_d - z)), \tag{4.1}$$

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$$M(z,\kappa) = \frac{2\kappa}{(2-\kappa)^2} - \frac{4\kappa(2+\kappa)}{(2-\kappa)^3}\sqrt{z_d - z} + \mathrm{O}((z_d - z)) \tag{4.2}$$

and

$$H_0(z,\kappa,1) = \frac{\kappa}{4(2-\kappa)^2}\left(\frac{1}{z_d - z}\right) + \mathrm{O}\left(\frac{1}{\sqrt{z_d - z}}\right), \tag{4.3}$$

for $z \to z_d^-$.

For $\kappa = 2$ one can expand around the singularity at $z_d = 1/2$ to find

$$G_0(z,2,1) = \frac{1}{2\sqrt{z_d - z}} - 1 + \mathrm{O}((z_d - z)), \tag{4.4}$$

$$M(z,2) = \frac{1}{4(z_d - z)} + \mathrm{O}\left(\frac{1}{\sqrt{z_d - z}}\right) \tag{4.5}$$

and

$$H_0(z,2,1) = \frac{1}{32(z_d - z)^2} + \mathrm{O}\left(\frac{1}{(z_d - z)}\right), \tag{4.6}$$

for $z \to z_d^-$.

For $\kappa > 2$ one can expand around the singularity at $z_a = \sqrt{\kappa - 1}/\kappa$ to find

$$G_0(z,\kappa,1) = \frac{\kappa - 2}{2\kappa\sqrt{\kappa - 1}}\left(\frac{1}{z_a - z}\right) + \mathrm{O}(1), \tag{4.7}$$

$$M(z,\kappa) = \frac{(\kappa-2)^2}{4(\kappa-1)\kappa^2}\left(\frac{1}{(z_a - z)^2}\right) + \mathrm{O}\left(\frac{1}{(z_d - z)}\right) \tag{4.8}$$

and

$$H_0(z,\kappa,1) = \frac{1}{4\kappa}\left(\frac{1}{(z_a - z)^2}\right) + \mathrm{O}\left(\frac{1}{(z_d - z)}\right), \tag{4.9}$$

for $z \to z_a^-$.

The duplicate singularities at $z = -z_s$ imply that the partition functions are zero when n is odd. They also imply that at even n the asymptotic values of the partition functions as n becomes large are double what they would be if calculated by considering only $z = z_s > 0$.

Using Darboux's lemma (see chapter 5 of [20]) and elementary Taylor series one finds for $0 < \kappa < 2$ and n even as $n \to \infty$ that

$$Z_n^{(0)}(\kappa,1) = [z^n]G_0(z,\kappa,1) = \left[\frac{2\kappa}{(2-\kappa)^2}\sqrt{\frac{2}{\pi}}\right] 2^n n^{-3/2}(1 + \mathrm{O}(n^{-1/2})), \tag{4.10}$$

$$V_n(\kappa) = [z^n]M(z,\kappa) = \left[\frac{2\kappa(2+\kappa)}{(2-\kappa)^3}\sqrt{\frac{2}{\pi}}\right] 2^n n^{-3/2}(1 + \mathrm{O}(n^{-1/2})) \tag{4.11}$$

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and

$$S_n^{(0)}(\kappa,1) = [z^n]H_0(z,\kappa,1) = \left[\frac{\kappa}{(2-\kappa)^2}\right] 2^n(1+\mathrm{O}(n^{-1/2})), \tag{4.12}$$

while for $\kappa = 2$ and n even we have that

$$Z_n^{(0)}(2,1) = [z^n]G_0(z,2,1) = \sqrt{\frac{2}{\pi}}2^n n^{-1/2}(1+\mathrm{O}(n^{-1/2})), \tag{4.13}$$

$$V_n(2) = [z^n]M(z,2) = 2^n(1+\mathrm{O}(n^{-1/2})) \tag{4.14}$$

and

$$S_n^{(0)}(2,1) = [z^n]H_0(z,2,1) = \tfrac{1}{4}2^n n(1+\mathrm{O}(n^{-1/2})). \tag{4.15}$$

Finally for $\kappa > 2$ and n even we have that

$$Z_n^{(0)}(\kappa,1) = [z^n]G_0(z,\kappa,1) = \left[\frac{(\kappa-2)}{(\kappa-1)}\right]\left(\frac{\kappa}{\sqrt{\kappa-1}}\right)^n (1+\mathrm{O}(n^{-1})), \tag{4.16}$$

$$V_n(\kappa) = [z^n]M(z,\kappa) = \left[\frac{(\kappa-2)^2}{2(\kappa-1)^2}\right]\left(\frac{\kappa}{\sqrt{\kappa-1}}\right)^n n(1+\mathrm{O}(n^{-1})) \tag{4.17}$$

and

$$S_n^{(0)}(\kappa,1) = [z^n]H_0(z,\kappa,1) = \left[\frac{\kappa}{2(\kappa-1)}\right]\left(\frac{\kappa}{\sqrt{\kappa-1}}\right)^n n(1+\mathrm{O}(n^{-1})). \tag{4.18}$$

The results for the partition function scaling in equations (4.10), (4.13) and (4.16) have appeared previously [21].

These lead immediately to our final results.

5. Results and conclusions

Our final results are that for $\kappa < 2$ and n even, the average height, $\mathcal{R}(n,\kappa)$, is

$$\mathcal{R}(n,\kappa) = \sqrt{\frac{\pi}{8}}n^{1/2} + \mathrm{O}(1), \tag{5.1}$$

while for $\kappa = 2$ and n even

$$\mathcal{R}(n,\kappa) = \sqrt{\frac{\pi}{32}}n^{1/2} + \mathrm{O}(1). \tag{5.2}$$

For $\kappa > 2$ and n even we have

$$\mathcal{R}(n,\kappa) = \frac{\kappa}{2(\kappa-2)} + \mathrm{O}(n^{-1}). \tag{5.3}$$

That is, $\mathcal{R}(n,\kappa)/n^{1/2}$ takes on a constant value for $\kappa < 2$ and jumps discontinuously to a smaller constant when κ becomes 2. It is unexpected that the average height for $\kappa < 2$ does not depend on κ.

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For completeness we also give that for $\kappa < 2$ and n even, the average number of visits to the surface, $\hat{m}(n, \kappa)$, is

$$\hat{m}(n, \kappa) = \frac{2 + \kappa}{2 - \kappa} + \mathrm{O}(n^{-1/2}), \tag{5.4}$$

while for $\kappa = 2$ and n even

$$\hat{m}(n, 2) = \sqrt{\frac{\pi}{2}} n^{1/2} + \mathrm{O}(1). \tag{5.5}$$

Finally,

$$\hat{m}(n, \kappa) = \frac{\kappa - 2}{2(\kappa - 1)} n + \mathrm{O}(1) \tag{5.6}$$

for $\kappa > 2$ and n even. Note that, as opposed to the average height, the average number of visits does vary with κ for all κ, including for $\kappa < 2$.

We have calculated the asymptotics of the average height of Dyck paths from a wall as a function of the wall potential parametrized by the Boltzmann weight κ. The average height is simply related to the average area between the polymer and the wall. We have approached the calculation via generating functions that require us to extend the Temperley method to coupled systems of difference equations. A motivation has been current work on entanglement in far from equilibrium stationary states. We can see from our current calculation that neither for $\kappa < 2$ nor at the critical point, at $\kappa = 2$, does the *valence bond entanglement entropy* develop logarithmic corrections as would be predicted by the theoretical prediction in equation (1.3). However, other measures of entanglement such as 'mutual information' and 'boundary Shannon entropy' [1] do display logarithmic behaviours it seems for this model. This difference is our main conclusion from the calculation with regard to entanglement estimators.

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