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Ordered cellular automata in one dimension

P.-M. Binder, D. Y. K. Ko, A. L. Owczarek and C. J. Twining

Department of Physics, Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, G.B.

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Abstract. — We study a probabilistic one-dimensional majority-rule two-state cellular automaton and examine the stability of ordered magnetised states in systems of size L as the neighbourhood radius R varies. We find that a scaling $R \sim \ln L$ is sufficient for an ordered phase to be metastable, i.e., to survive for times much longer than the typical critical fluctuation. The lattice magnetisation obeys a scaling relation which agrees with results from mean-field analysis.

Probabilistic cellular automata (PCA) have been of much recent interest largely as a result of their wide ranging applications. For example, they are paradigms for multi-component computing and information storage structures [1, 2, 3]. They are also relevant to models of non-equilibrium dynamics in physical systems [4] as well as having direct connections to equilibrium statistical mechanics in higher dimensions [5, 6] (for reviews and extensive bibliography, see [7, 8, 9]). In many of these applications the question of interest is the existence of stable macroscopically ordered states in the presence of noise which disrupts the system in a manner similar to the effects of temperature in equilibrium statistical mechanical systems. In contrast to the usual in statistical mechanics one-dimensional PCA in general do not obey detailed balance and hence do not have equivalent *one-dimensional* Hamiltonians. They can, however, usually be mapped to highly anisotropic spin systems in higher dimensions, with properties which may be significantly different, as was the case for the Toom model [1].

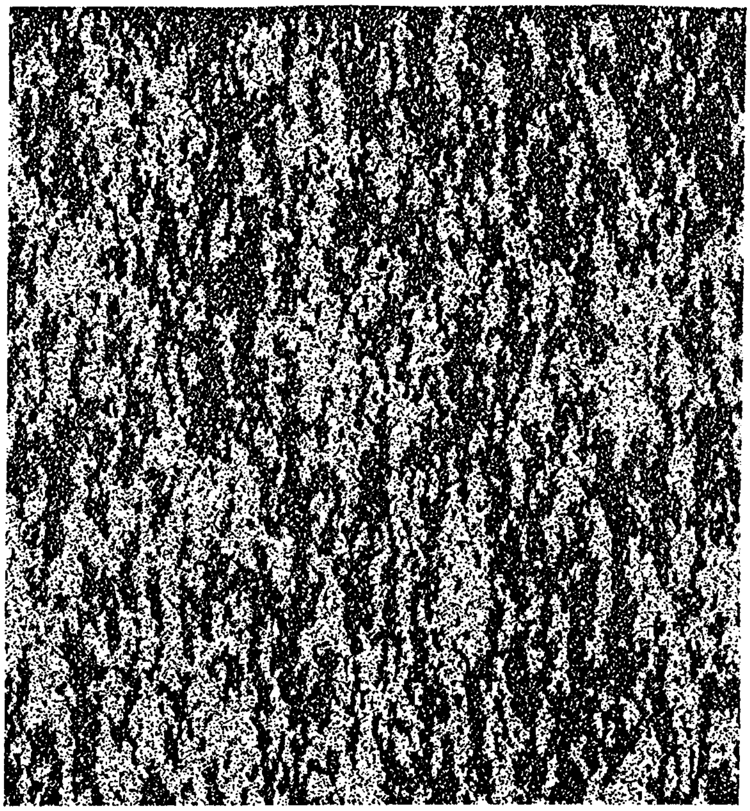
There is a well-known free-energy argument [10] against order in one dimensional spin systems at finite temperature, indicating that when interactions are short-range phases tend to mix in arbitrarily small segments, and therefore there is no long-range order (for discussions of order in long-range, one-dimensional spin systems, see [11, 12]). The above argument does not necessarily apply to PCA, although a belief is that finite one-dimensional PCA with positive local transition rates are ergodic and all states are therefore eventually unstable to noise. This is known as the "positive rate conjecture" (see for example [13] for a discussion of the continuous-time case). By way of an counterexample, however, a complex hierarchical PCA possessing an ordered state in the presence of noise was proposed by Gacs [3].

Even if finite systems are ergodic and have a unique stationary state, there may be states which are *metastable*, i.e., the probability of the system leaving them decreases roughly exponentially with system size (see [14] and Refs. therein). In this letter we examine the metastability of states initially magnetised, in the particular case where the interaction range is allowed to scale with the system size. We study numerically a simple 1d PCA model, and show that an interaction range scaling as $R(L) \sim \ln L$ is sufficient to produce a "macroscopically" ordered state which will survive noise for times much greater than the time fluctuations observed in the system when it appears to be critical. For a macroscopic system ($L \sim 10^7$ in one-dimension) this indicates that a microscopic interaction neighbourhood ($R \sim 16$) is sufficient for metastable order. We present also a mean field theory which explains the behaviour of the system at low noise levels.

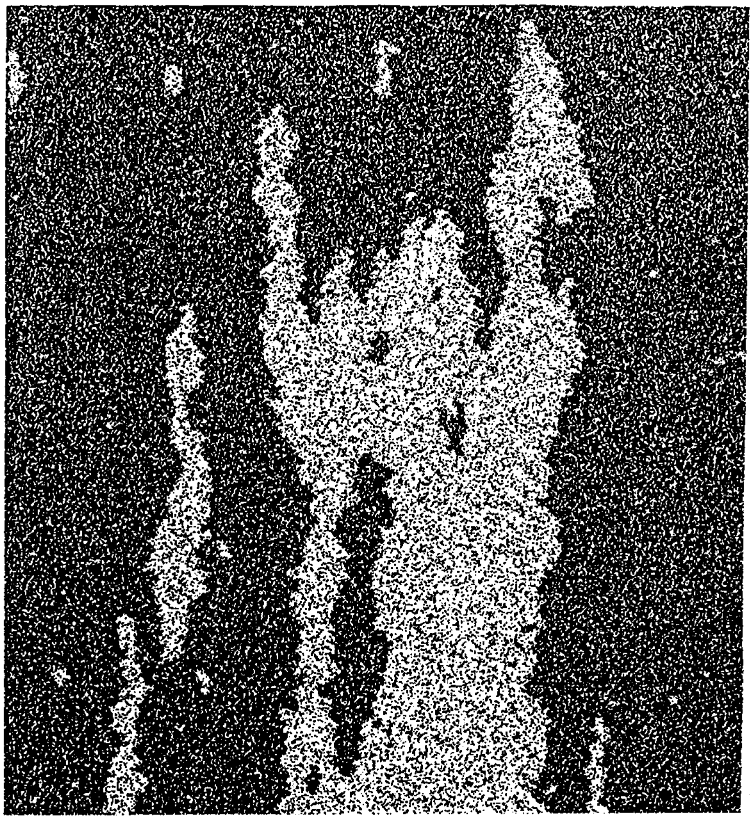
The rule we study is as follows: we have a line of L sites with periodic boundary conditions. Each site can take on values ± 1. At each time step a local field is first summed for each site, over the site itself and R neighbours on each side, then all sites are updated simultaneously to be $+1$ with probability $\frac{1}{2}+p$ if the local field is positive, or $\frac{1}{2}-p$ if the local field is negative. Therefore p measures the noise in the system and the limiting values are $p=0$, corresponding to completely random, uncorrelated updating, and $p=\frac{1}{2}$, corresponding to deterministic majority rules (the $R=1$ case has been studied analytically in [15]). The corresponding probabilistic case for $R=1$ has been studied in considerable detail by Gray [16]. The transition probability to a $+1$ state *versus* local field is thus a step function, reminiscent of spin Hamiltonians at low temperatures. However, since it is not exactly of the form of a hyperbolic tangent [17] our system does not satisfy detailed balance and hence does not possess a one-dimensional Hamiltonian.

In figure 1 we show typical space-time pictures of the model. Time evolves from top to bottom; plus sites are represented by black pixels. We choose to start all lattice nodes with a value $+1$. The lattice size is 720, the number of time steps shown is 800 and $p=0.35$. We see how, as we increase R, the system goes from (I) a regime of many small domains with average zero magnetisation to (II) a regime of very few large, long-lived domains in which the overall magnetisation could well be positive or negative to (III) a mostly-magnetised quasi-homogeneous regime in which the minority sites are distributed at random. For sufficiently low values of p (III) the small-R behaviour becomes less structured than (I), but still has zero average magnetisation. For lower p, as R is increased, regime (II) is increasingly difficult to find and the system apparently changes from regime (I) to regime (III) directly. (IV) also shows an unmagnetised structureless regime for $R=2$ and $p=0.1$.

Before we discuss the results reported below several points need to be noted. The first is that we measure the *absolute* value of the magnetisation at each time step, since in regime (II) the overall magnetisation could equally be positive or negative. This choice still allows us, however, to distinguish between the homogeneous, the multiple-domain and the noisy regimes. A second point is that an initial condition of all sites magnetised to $+1$ has been chosen. This choice is clearly more appropriate than an initial condition of randomly magnetised sites since we plan to investigate the stability of the *ordered* phases. Surprisingly, due to the synchronous updating we find that the mean-field magnetisation is reached in one time step. Furthermore, the arguments given later in this letter apply to the stability of a single low-temperature (or low-noise) ground state. We have also found that at very low noise levels and large enough R it takes a random initial condition many iterations to evolve to a metastable ordered state, so that simulation becomes prohibitively long. Finally, we note that a large number of sampling time steps are required to reproduce good data. While increasing R for fixed L there exists a maximum in the size and length of the magnetisation fluctuations as the system moves towards the magnetised state. We believe that this maximum is an indication of critical behaviour in the system. At these "critical" points we have examined both the initial relaxation and fluctuation times. The relaxation time, as defined in the concept of damage healing [18], is how long it takes

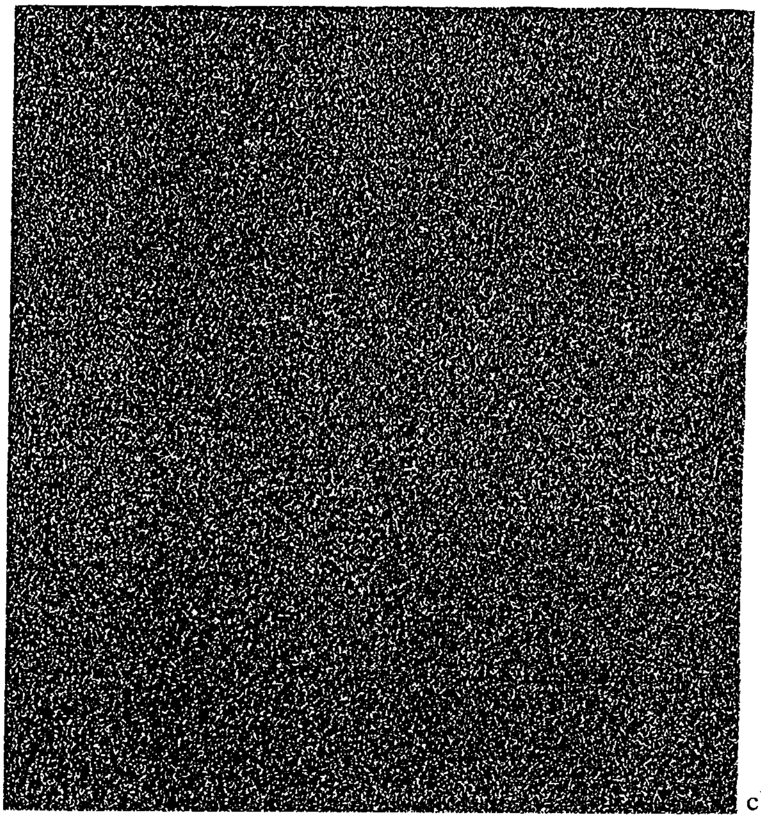


a)



b)

Fig.1. — Phenomenology of the model: (I) unmagnetised regime ($R = 1$, $p = 0.35$); (II) multiple domain regime ($R = 5$, $p = 0.35$); (III) homogeneous magnetised regime ($R = 11$, $p = 0.35$); (IV) Low-p unmagnetised structureless regime ($R = 2$, $p = 0.1$).



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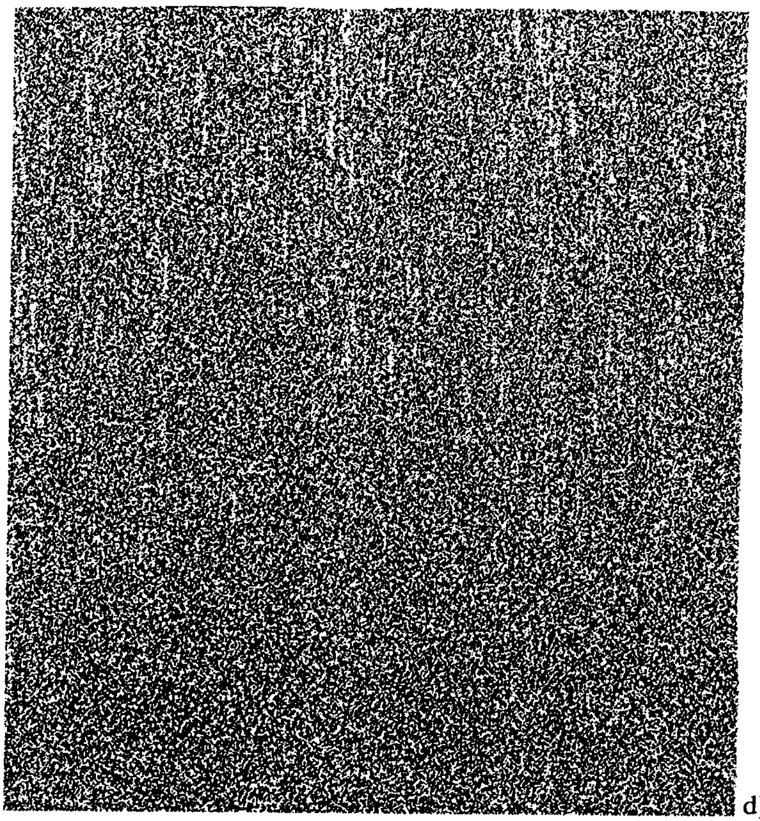

d)

Fig.1. — (continued)

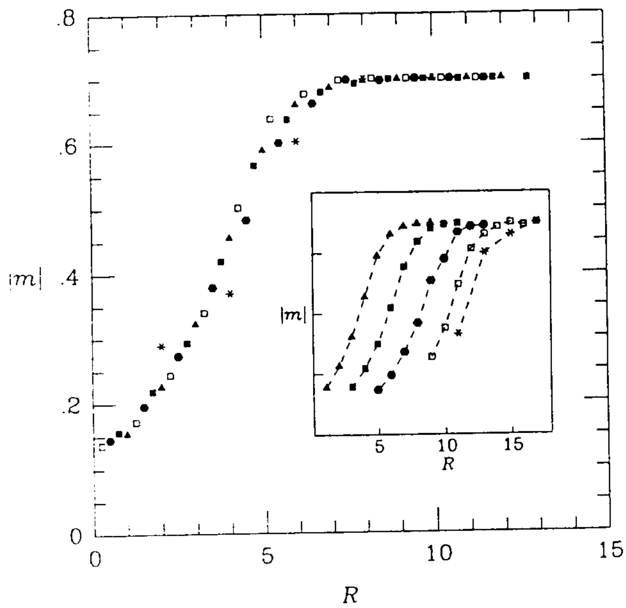

Fig.2. — Absolute magnetisation *versus* R for $64 \leq L \leq 16384$ scaled together according to (1) for $p = 0.35$. The inset shows the unshifted curves.

for two systems with different initial conditions, evolving under the same random numbers, to evolve to nearly identical states. The fluctuation time is defined by the dominant wavelength of the magnetisation-time history, and was estimated visually. Our choice of initial and measuring times are greater than the relaxation and the fluctuation times respectively. This is crucial, as it indicates that the magnetised state survives over many times this well-defined time scale. Depending on the lattice sizes and the noise, the results investigated in this letter have been sampled over 2×10^5 to 2×10^7 time steps.

In figure 2 we present the time averaged absolute magnetisation $|m|$ of the lattice for $L =$ 64, 256, 1024, 4096 and 16384 and for a noise level of $p = 0.35$. The curves for different system sizes have been shifted by multiples of 1.59 showing that the magnetisation satisfies a scaling relation of the form

$$|m(R,L)| = f(R - A \ln\ L) \tag{1}$$

where $A = 2.2/\ln 4 = 1.59$. The inset shows the original unshifted data. For large enough values of R the system clearly remains magnetised and the existence of a clearly magnetised state is observed even for a noise level of $p = 0.2$. We note that for a given p the entire $|m(R,L)|$ curve scales into the same universal form rather than just the crossover point scaling together. This extensive scaling of the entire curve holds for values of p down to ~ 0.2 before the functional form for different L begins to deviate. For various values of p, and for smaller lattice sizes ($L \leq 1024$), we have also measured $A(p)$; the results are given in figure 3.

We can understand some aspects of the behaviour of the magnetisation in terms of a simple analysis. The high-p (low-noise) limit corresponds to the situation of having a single domain across the entire lattice. In this situation the local magnetisation within a domain can be predicted using a mean field approximation (see [1] and Refs. therein). Briefly, consider the interior of a domain where the minority sites are essentially uncorrelated (regime III) and have a density $\rho = (1 - m)/2$. The probability of minority sites being a majority in a particular

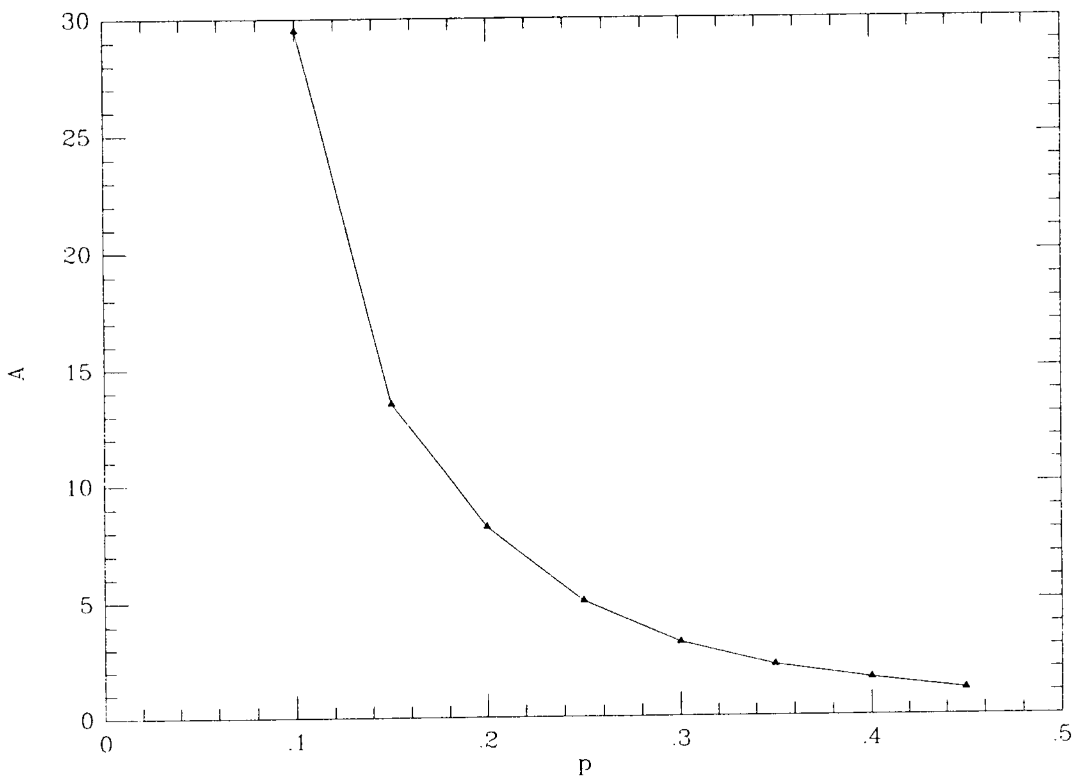

Fig.3. — Amount of shift $A(p)$ *versus* p necessary for scaling of absolute magnetisation curves.

neighbourhood is

$$f(\rho) = \sum_{n=R+1}^{2R+1} \binom{2R+1}{n} \rho^n (1-\rho)^{2R+1-n}. \tag{2}$$

Updating this neighbourhood we get a self-consistency condition relating ρ to p

$$\rho = 2p\, f(\rho) + \frac{1}{2} - p\ . \tag{3}$$

In the large-p (or small-ρ) limit this reduces to $\rho \sim \frac{1}{2} - p$ such that in the large-R limit the magnetisation becomes

$$|m| = m_{\rm mf} \rightarrow 2p\ . \tag{4}$$

This value for the magnetisation agrees with the simulations down to surprisingly low values of p, around 0.2.

In figure 4 analytic results for ρ along with simulation measurements of the magnetisation within domains for $R = 20$ are shown. The agreement is very good for most values of p, almost down to the transition point between regimes (I) and (II). We note, all the same, that the mean field theory ignores the dynamics of the domain evolutions and should therefore not be interpreted as a proof of the stability of the ordered phase. However, it does provide a remarkably good prediction to the magnetisation within domains, and further predicts the value of p at which the onset of regime (I) occurs for a given R reasonably well.

The logarithmic scaling of the critical interaction neighbourhood may be understood by considering the probability of a metastable ordered state evolving to a similar state with a large enough fluctuation of the opposite magnetisation. Consider two systems of lengths L_1

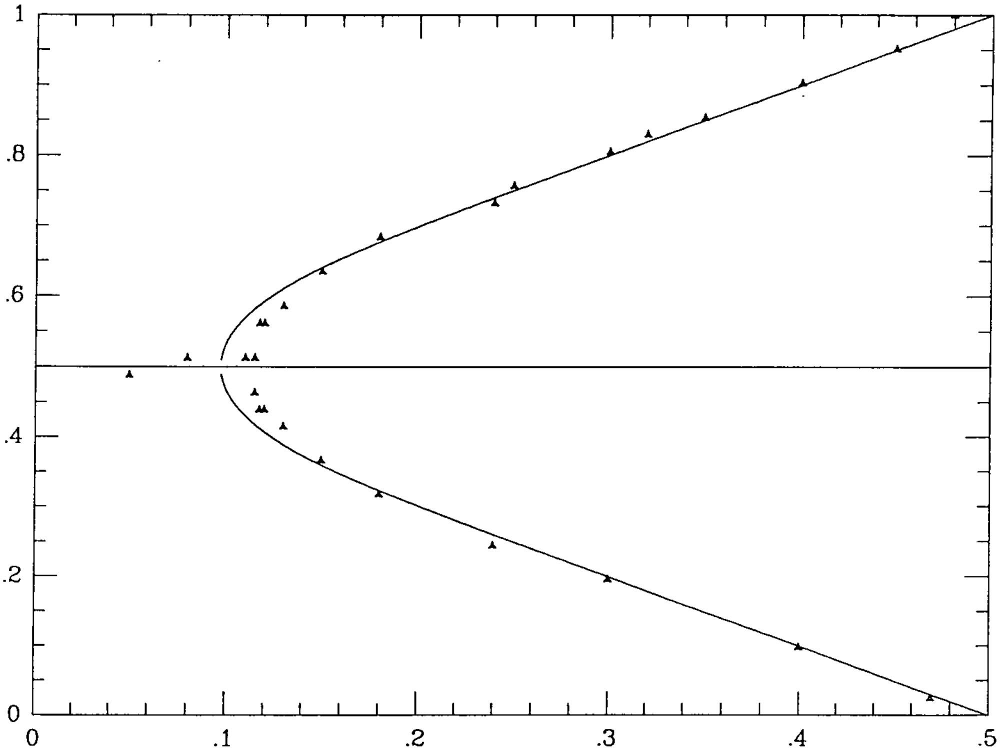

Fig.4. — Mean-field theory estimate (solid line) and simulations (points) for the density of majority sites within domains for $R = 20$ over the full range of p.

and L_2, initially with $m \sim m_{\mathrm{mf}}$, with neighbourhood radii R_1 and R_2. We examine the condition on R under which domains of size R_1 and R_2, which should survive for at least a few time steps, will form in both systems in a single time step. We consider in particular the case of low level of noise (p close to $\frac{1}{2}$). The number of sites that must change is mR_1 and mR_2 respectively. Equating a simple estimate for the probabilities of such small domain formation in the two systems gives

$$L_1\, (\frac{1}{2} - p)^{mR_1} = L_2\, (\frac{1}{2} - p)^{mR_2}. \tag{5}$$

This is satisfied if the interaction neighbourhood is

$$R(L) = A(p) \ln L + B \tag{6}$$

where

$$A(p) = -\frac{1}{2p \ln(\frac{1}{2} - p)}, \tag{7}$$

that is, if the interaction neighbourhood radius scales as the logarithm of the system size. Clearly, for high-p (low-noise) the dynamics of domain formation will be important and may not be ignored in the manner discussed here. However, it is not unreasonable to expect a more detailed argument to still yield a logarithmic scaling of R with L.

In summary, in this letter we have presented numerical data and analytic arguments which show that an interaction neighbourhood scaling as the logarithm of the system size is sufficient to maintain order for fairly long times in metastable magnetized states of one-dimensional PCA in the presence of considerable levels of noise. Even if the differences between cellular automata and spin systems can be considerable, as discussed in the introductory paragraph, the results in this letter are consistent with a scaling relation recently derived for order at non-zero temperatures in finite one-dimensional spin systems with a well-defined free energy [19].

The results are pertinent to the question of order in real one-dimensional systems, since physical systems are always finite. They suggest that a microscopic interaction neighbourhood of the order of $\sim 10-20$ sites alone may be sufficient to maintain a macroscopic system in an ordered state for a long time even when significant levels of noise are present.

Several aspects of our system need further understanding. These include a full analysis of the functional form of $A(p)$, the transient behaviour when the system starts from a random initial condition far away from equilibrium, and explanations of the fact that the whole function $|m(R)|$ scales so well over a large range of radii. Much of this will involve analysis of the dynamics of the domain formation, and the "damage healing" from initial conditions; work on these aspects is in progress. Furthermore, the stability properties of configurations in regime (II), with several large magnetised domains, are unknown even for low noise.

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