J. Phys. A: Math. Gen. 25 (1992) L21–L24. Printed in the UK

LETTER TO THE EDITOR

Cut-off scalings for one-dimensional order

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Received 3 September 1991

Abstract. We consider one-dimensional Ising systems of length L with an interaction cut-off $R(L)$ in the limit $R \to \infty$. We present free energy arguments for order at low temperatures provided $R(L) > (A \ln L)^{1/\kappa}$, where κ is a new exponent; this opens the possibility of one-dimensional pseudo-order in macroscopic systems.

It has been a long time since van Hove (1950) proved that one-dimensional models in statistical mechanics with finite-ranged two-particle interactions do not have phase transitions at non-zero temperatures. Since then, many one-dimensional models have been studied (Lenard 1961, Baxter 1963, 1964, 1965, Lieb and Matthis 1966, Joyce 1966, Kac and Thompson 1969). It has been shown that one-dimensional systems with strong enough long-ranged two-particle forces may have such transitions in the thermodynamic limit. Ruelle (1968) and Dyson (1969) have proved several theorems concerning one-dimensional lattice systems with two-particle interaction strength $J(r)$ at a distance r. It is now generally believed that Ising chains with interactions $J(r) = Ar^{-a}$, $1 < a \leq 2$, do possess a critical point at non-zero temperature. The critical behaviour is of the mean-field variety for $a < 3/2$. When the interactions have a finite cut-off, decay exponentially or algebraically with $a > 2$, no finite-temperature transition occurs.

In this letter we study Ising chains with interactions of finite range R in the large-L limit where R scales with L, the system size. We use simple Landau-type arguments to find out how this scaling must be for order to exist at low but non-zero temperatures. We find this scaling to be *logarithmic*, which opens up the possibility of macroscopic one-dimensional systems with effective order caused by interactions of small but finite range. This large-L limit is not the usual thermodynamic limit. Nevertheless, the free energy per particle may still exist (the free energy is extensive) if the interaction strength is also scaled. This type of scaling is reminiscent of the Kac potential (Lebowitz and Penrose 1966), although it is not the same.

Consider an Ising chain of length L with Hamiltonian

$$\mathscr{H} = -\sum_{i=1}^{L} \sum_{r=1}^{R} J(r) \sigma_i \sigma_{i+r} \tag{1}$$

where the indices are understood to be modulo L, and $J(r) \geq 0$ for all r.

We wish to find under what conditions an ordered phase is stable to the formation of domain walls. In this regard, we consider the free energy difference between a completely ordered state and one in which this order has been broken into two regions by the formation of one domain. In particular, we investigate the following interaction potentials:

$$J(r) = Jr^{-a} \qquad 0 < a < \infty \tag{2a}$$

$$J(r) = J \qquad r \leqslant R \tag{2b}$$

and

$$J(r) = J\delta_{r,R} + \varepsilon\delta_{r,1} \tag{2c}$$

where $J(r) = 0$ for $r > R$ in all three cases. In model (2c) we have added a small nearest-neighbour interaction $\varepsilon > 0$ to prevent the system from decoupling.

The free energy difference ΔF between the two-phase system and the ordered system at low temperature can be estimated by considering both the energy and the entropy of the interfaces. The entropy of the two interfaces is approximately

$$\Delta S \sim \ln[L(L-2R)] \sim 2 \ln L \tag{3}$$

where $L(L-2R)$ is the number of ways two interfaces can be placed a distance at least R apart in a ring of length L, forming a stable domain.

The energy is calculated from the number of broken bonds across the interfaces. This is in each case $\Delta E \sim 2 \Sigma J(r)r$. Hence,

$$\Delta E \sim 2JR^{2-a} \qquad (a < 2)$$

$$\Delta E \sim 2J \ln R \qquad (a = 2)$$

and

$$\Delta E \sim \text{constant} - 2JR^{2-a} \qquad (a > 2) \tag{4a}$$

for the models (2a),

$$\Delta E \sim 2JR^2 \tag{4b}$$

for model (2b), and

$$\Delta E \sim 2JR \tag{4c}$$

for model (2c), ignoring the small contribution from the ε term.

To ensure the stability of the ordered phase, we require that ΔF be positive, that is,

$$\Delta E > T\Delta S. \tag{5}$$

Substituting the respective terms into (5) yields the scalings

$$R(L) > \left[\frac{T}{J} \ln L\right]^{1/(2-a)} \qquad (a < 2)$$

$$J \ln R(L) > T \ln L \qquad (a = 2) \tag{6a}$$

$$\text{constant} - \frac{2J}{R(L)^{a-2}} > 2T \ln L \qquad (a > 2)$$

$$R(L) > [(T/J) \ln L]^{1/2} \tag{6b}$$

and

$$R(L) > (T/J) \ln L \tag{6c}$$

for all the models described in equation (2).

The first point to notice is that for $a > 2$ there is no way to scale R to make an ordered phase stable. This agrees with earlier findings that models with algebraic interactions which decay faster than quadratically do not have phase transitions. Secondly, apart from the case $a = 2$ in model (2a), all the scalings $R(L)$ are logarithmic. For model (2c) and model (2a) with $a = 1$ (which is also the marginal case for a thermodynamic limit to exist if R is made large before L), the scaling is purely logarithmic. For model (2b) the scaling is very slow—the square root of a logarithm. Since $\ln(10^7) \sim 16$ and $[\ln(10^7)]^{1/2} \sim 4$ are not too large, we conjecture that there is a possibility of some macroscopic system modelled by one of these Ising Hamiltonians to exhibit long-range order at low temperatures. We expect this order to be usually of the mean-field type (non-trivial long-range behaviour is not discounted). A final point is that in the large-L limit $R \ll L$; the last approximation in (3) is then justified, and the results are self-consistent in this sense.

All the scalings, except $a = 2$ for model (2a), can be put into the form $R(L)^\kappa > A \ln L$, with the exponent κ depending on the model in question. We expect this logarithmic scaling with system size to be general. Non-Hamiltonian systems such as cellular automata (CA) should exhibit this behaviour, and should be more suitable from a simulational point of view to test the ideas in the present letter. Also, higher dimensional systems such as the two- and three-dimensional versions of the Hamiltonian (1) should have analogous behaviour, where the crossover is now interpreted not as between order and disorder, but between long-range dominated and the usual fluctuation-dominated criticality.

Lastly we consider whether the free energy of these systems can be made extensive by rescaling the interaction strength with L. We concentrate on model (2b). The total energy of the ordered phase is

$$\mathscr{E} = -\sum_{i=1}^{L}\sum_{r=1}^{R} J = JRL. \tag{7}$$

By scaling $J(R)$ as

$$J(R) = \frac{J_1}{R} \tag{8}$$

the free energy of the ordered state is rendered extensive. It is important to note that this scaling does not destroy the possibility of order at low temperatures by again scaling R with L. Substituting (8) into (6a) we have that the relationship

$$R(L) > \frac{T}{J_1} \ln L \tag{9}$$

is required to achieve a stable ordered state, and with (8) to give an extensive free energy in this case. Note that (9) is stronger than (6b), which was derived without the extensivity condition (8). These two requirements are not mutually exclusive.

In conclusion, we have provided simple but testable arguments for the existence of order in one-dimensional systems when the range of interactions is scaled with the system size, and discussed the possibility of extensivity in this limit. This study is also a first attempt at bridging the gap between the complementary techniques of finite size (Barber 1983) and finite range scaling (Novotny *et al* 1986, Glumac and Uzelac 1989) in the study of infinite-range thermodynamic systems. One goal of further work could be a full scaling theory of long-range systems to include system size, interaction range and temperature.

We thank Dr C J Twining and Professor D Sherrington for helpful discussions, Professor V Privman for pointing out some references, Dr R B Stinchcombe for comments on the manuscript, and the SERC for financial support.

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