Correlation Function for the Baxter Model

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We derive an exact expression for the low-temperature correlation function in the eight-vertex-equivalent Ising model. We provide a phenomenological interpretation of the qualitative nature of the transfer-matrix spectrum in terms of bubble excitations and speculate about the behavior in nonzero magnetic field.

The planar symmetric eight-vertex, or Baxter, model in which arrows are assigned to every edge of a quadratic lattice such that the number of arrows pointing towards each vertex is even, and the vertex weights are invariant under reversal of all arrows, is known to be equivalent to an Ising magnet; here, spins on the dual lattice are coupled by next-nearest two-body interactions and nearest-neighbor four-body interactions, but there is no external field.[1] The free energy of this model, obtained by transfer-matrix diagonalization,[2] led to a refinement of the concept of universality to handle the continuous dependence of critical exponents on interaction parameters, an astonishing feature to emerge from the calculation. Johnson, Krinsky, and McCoy[3] (JKM) extended Baxter's work to obtain bands of excitations of the transfer matrix (hereinafter denoted V) nearest to the maximal eigenvalue Λ_0. These results were recaptured recently and extended to the entire spectrum using a matrix-inversion technique.[4] JKM obtained the correlation length and associated critical exponent; this is a subtle exercise since, although V is normal, it is not self-adjoint and has complex eigenvalues. Another interesting phenomenon noted by JKM was the emergence of single-particle bound states from the two-particle continua as the parameters vary in the ferromagnetically ordered phase. Later in this Letter, we shall introduce a much simpler bubble model[5] in which a similar spectral

change is associated with a pinning-depinning transition. This comes about because the four-body interaction can induce an attraction between opposite sides of the bubble causing its collapse at low enough temperatures into a thread of finite thickness, a manifestation of the one-particle bound-state formation.

Almost nothing is known exactly about the decay of the pair-correlation functions in these models (beyond the correlation length). In the one-particle region established by JKM, we give a new simple argument which extracts the Ornstein-Zernike behavior directly from the JKM excitation energies. We then show that the bubble model results are consistent with this and deduce the correlation behavior between the spectral transition temperature and the critical temperature, thereby establishing the existence of an intermediate phase. The works of Baxter[2] and JKM showed that the change in character of the spectrum has *no thermodynamic consequences*, even though the correlation length is singular. Our phenomenological model suggests that this picture may be quite common.

On an infinite-length cylinder with axis in the transfer direction (1,0) standard theory gives the pair-correlation function for local observables $\hat{O}(y)$ in each column as

$$\rho(x,y) = \langle \phi_0 | \hat{O}(0)(\Lambda_0^{-1}V)^x \hat{O}(y) | \phi_0 \rangle, \tag{1}$$

where V is the transfer matrix with unique maximum

eigenvector $|\phi_0\rangle$ and eigenvalue Λ_0. The lattice geometry implies $[V,T]=0$, where T is the unit translation in the (0,1) direction. Thus V and T have common eigenvectors $|\phi_j\rangle$ with eigenvalues $\Lambda_0 \exp[-\gamma(j)]$ and $\exp[i\omega(j)]$, respectively. Equation (1) can then be written as

$$\rho(x,y)=\sum_j |\langle\phi_0|\hat{O}(0)|\phi_j\rangle|^2$$
$$\times \exp[-|x|\gamma(j)+iy\omega(j)]. \qquad (2)$$

The problem of obtaining correlation functions via V is to get the matrix elements, only accomplished so far for the planar Ising model.[6] We now show one situation where a reasonable hypothesis allows us to give a general method which we then apply to the Baxter model.

Suppose the sum over the $|\phi_j\rangle$ can be resolved into a sum over n-particle bands. This means the $|\phi_j\rangle$ can be labeled by $|\phi(\omega)_n\rangle$, where

$$V|\phi(\omega)_n\rangle=\exp\left(-\sum_1^n \gamma(\omega_j)\right)|\phi(\omega)_n\rangle \qquad (3)$$

and

$$T|\phi(\omega)_n\rangle=\exp\left(i\sum_1^n \omega_j\right)|\phi(\omega)_n\rangle. \qquad (4)$$

Such a grouping is known to be possible for the normal 2D Ising model—there, the problem is establishing (4)—and it certainly holds for the first few bands at least of the Baxter model, as shown by JKM. In this case, (4) is obtained by varying a parameter, which does not occur in $|\phi_j\rangle$ since it labels a commuting family of V, to a special value for which V is itself a translation. In general, (4) arises from an underlying plane-wave structure $\exp(i\sum_1^n y_j\omega_j)$, where y_j are particle positions; Bethe's insight[7] was that the ω_j do not have to satisfy $\exp(iN\omega_j)=1$, so long as their sum does. Furthermore, they do not have to be real provided they occur in complex-conjugate pairs which produce bound states out of the plane waves. Neither do the scattering phase shifts between such plane waves have to be ± 1 in value.

Let us group the terms in the sum (2) by particle number n (sometimes called a dispersion series) and denote the nth term by $\rho_n(x,y)$. Then the $n=0$ term gives the contribution (*long-range order*).[2] If the lattice is symmetric under rotation by $\pi/2$ about the perpendicular axis, then $\rho(x,0)=\rho(0,x)$ (the thermodynamic limit having been taken). Since $T\neq T_c$, $\min \mathrm{Re}\gamma(\omega)>0$; thus the least n value dominates the spatial decay of $\rho(x,0)$. We assume the same to be true for $\rho(0,x)$ and moreover that $\rho_1(x,0)=\rho_1(0,x)$ for all x. In support of this, for the usual planar Ising ferromagnet, it is known that $\rho_n(x,0)=\rho_n(0,x)$ for all x and for all n.[8] Let us define

$$K(\omega)=\lim N|\langle\phi_0|\hat{O}(0)|\phi(\omega)\rangle|^2.$$

Then our assumption with (2) gives

$$\int_0^{2\pi} d\omega\, K(\omega)\exp(ix\omega)$$
$$=\int_0^{2\pi} d\omega\, K(\omega)\exp[-|x|\gamma(\omega)]. \qquad (5)$$

Regarding this as the formula for the xth Fourier coefficient of $K(\omega)$, we invert, interchange order of integration and summation (easily justified), and sum a geometric series obtaining the homogeneous Fredholm equation

$$K(\omega_1)=\frac{1}{2\pi}\int_0^{2\pi} d\omega_2 K(\omega_2)\frac{\sinh\gamma(\omega_2)}{\cosh\gamma(\omega_2)-\cos\omega_1}, \qquad (6)$$

which of course may only have a trivial solution.

We now bring in some results of JKM: There is an $n=1$ band below T_c given, for an isotropic system, in terms of an elliptic variable u by

$$\exp[-\gamma(u)]=k\,\mathrm{sn}\left[\frac{u+ib}{2}\right]\mathrm{sn}\left[\frac{u-ib}{2}\right] \qquad (7)$$

and

$$\exp[i\omega(u)]=k\,\mathrm{sn}\left[\frac{u+ib+iK'}{2}\right]\mathrm{sn}\left[\frac{u-ib-iK'}{2}\right], \qquad (8)$$

where $[0,2\pi]$ for ω goes to $[0,4K]$ for u modulo periodicities. A new parameter ζ and elliptic modulus l are useful to define the modulus k of the elliptic functions in (7) and (8) in terms of the Ising interactions J and J_4. We have

$$l=\frac{1-\exp(-4J_4)/\sinh^2(2J)}{1-\exp(-4J_4)/\cosh^2(2J)} \qquad (9)$$

and $\mathrm{cn}(b,l)=\exp(-2J_4)/\cosh(2J)$. (The critical curve is given by $l=0$.) Finally, $b=3K(k)[2K(l)/3\zeta-1]$ and the new modulus k is defined by

$$K'(l)K'(k)=2\zeta K(k). \qquad (10)$$

It is natural to change variables in (6) to the elliptic ones, getting

$$J(u_1)=-\int_0^{4K} du_2 \frac{d\omega}{du_1}J(u_2)\frac{\sinh\gamma(u_2)}{\cosh\gamma(u_2)-\cos\omega(u_1)} \qquad (11)$$

with $J(u)=K(\omega(u))d\omega/du$. Certain structural analogies between (7) and (8) and Onsager's uniformization[9] can be exploited to solve (11). The solution, unique up to a multiplicative factor, is $J(u)=$const. Since

$$\rho^T(x,0)=A\int_0^{4K} du\exp[-x\gamma(u)], \qquad (12)$$

the Ornstein-Zernike (OZ) form[10,11]

$$\rho^T(x,0)\sim \exp(-\kappa x)/x^{1/2}$$

follows with $\kappa=-\ln[k/dn^2(b/2,k')]$; κ was originally

determined by JKM. Such a result would follow just from $J(u) > 0$ at the saddle point and sufficient analyticity for the contour deformation, if necessary, to the steepest-descent one. Our result $J(u)$ =const is much stronger and of such striking simplicity as to encourage further work. Equation (12) is still valid for the anisotropic system with suitable generalization of (7), as follows from the existence of commuting families of transfer matrices.

When there is no one-particle state, a two-particle band dominates (1) and (2), but our integral-equation technique does not suffice to extract the matrix element needed to derive prefactors in the asymptotic expansion. The correlation length is continuous across the boundary between these regimes, as shown by JKM, but has a jump in its first derivative.

The rest of this Letter is phenomenological in character. It is known that the bubble picture of excitations[5,12]

provides a powerful rationalization of the asymptotic behavior of $T < T_c$ correlation functions for the planar Ising model, including the case with nonzero magnetic field H.[5] In this picture, a spin-pair-correlation function differs from its limiting value of infinite separation because both spins lie inside a single closed contour, which delineates a bubble, and which separates the pair of points from the boundary. Since we are examining the system on a scale of the correlation length, small closed loops vanish but the average magnetization outside [inside] the bubble is $m(h,t)$ $[-m(h,t)]$, where $m(h,t)$ is the bulk value. The free energy of the matter inside and outside the bubble is given by Baxter's value when $H = 0$. The statistics of the bubble are controlled by the surface tension. The bubble is restricted to have only solid-on-solid (SOS) configurations with top and bottom given by $y_j^>$ and $y_j^<$, respectively, for $j = 0, \dots, L$. The probability of a configuration is

$$P_L(c) = \exp\left[2L\tau(0) + \sum_{j=1}^{L} 2K[(y_j^> - y_{j-1}^>)^2 + (y_j^< - y_{j-1}^<)^2] + 2hm(h,t)\sum_{j=0}^{L}(y_j^> - y_j^<) \right], \tag{13}$$

where $\tau(0)$ is the angle-dependent surface tension and $K = [\tau(0) + \tau^{(2)}(0)]/2$, a term the importance of which was stressed by Fisher, Fisher, and Weeks.[13] At the bulk critical point both τ and K vanish. The final term gives the effect of a magnetic field, proportional overall to the area of the bubble. The correlation function is

$$\rho^T(L,0) = m(h,t)^2 \sum_c P_L(c). \tag{14}$$

Such a model is also expected to be valid for the Ising model equivalent to the eight-vertex model since it is essentially a uniaxial magnet. The effect of J_4 is to introduce a contact interaction between the upper and lower perimeters of the bubble; this interaction is attractive for $J_4 > 0$. An extra factor

$$\prod_{1}^{L-1} [1 - a\delta(y_j^> - y_j^<)]$$

must be introduced on the right-hand side of (13). The sum over c in (14) is carried out using a transfer-integral technique (the y_j are taken continuous with $y_0^> = y_0^< = y_L^< = y_L^> = 0$ for simplicity). The transformation of coordinates $y_j^\pm = (y_j^> \pm y_j^<)/2$ separates the kernel in the transfer integral into center-of-mass and relative parts. The second is handled approximately by replacing $(y_j^- - y_{j-1}^-)^2$ by $|y_j^- - y_{j-1}^-|$ which gives an exactly diagonalizable kernel, described in detail elsewhere.[14] The salient features are that the center-of-mass kernel has a continuous spectrum on (0,1). The relative motion spectrum is more interesting.

(1) For $h = 0$, there exists a critical value of a, $a_c = 1/2K$. This in turn implies (for fixed energetic parameters) a pinning temperature T_p such that $a < a_c$ implies $T_p < T < T_c$. In this case there is again a continuous spectrum on (0,1) which gives the Kadanoff-Wu

(KW) anomalous form[15] $x^{-2}\exp(-\kappa x)$ when the center-of-mass contribution is multiplied in. As T decreases through T_p, a bound state comes out of the relative motion continuous spectrum. This is associated with the binding together of opposite sides of the bubble. Together with the center-of-mass term, this produces a one-particle band and OZ behavior (see Fig. 1).

(2) For $h > 0$, the behavior depends strongly on the $\text{sgn}(a - a_c)$: For $a < a_c$, as $h \to 0$,

$$\rho^T(x,0) \sim x^{-1/2}h\sum_{1}^{\infty}\exp[-2x\tau(1 + a^{-2/3}\gamma_j)], \tag{15}$$

where $\tilde{a} = [\tau(0) + \tau^{(2)}(0)]/h$ and $2^{1/3}\text{Ai}'(2^{1/3}\gamma_j) = \tilde{a} \times \text{Ai}(2^{1/3}\gamma_j)$ with scaled interaction $\tilde{a} = (a - a_c)a^{1/3}/a_c$ and $\text{Ai}(\cdot)$ is the Airy function. This type of series was extracted by McCoy and Wu[16] from their series expansion (a somewhat dangerous procedure) of $\rho^T(x,0)$ in powers of h for the planar Ising model below criticality. The behavior of the large-j terms, necessary for the integrability of $\rho^T(x,0)$ in x, is discussed elsewhere.[5,12]

As $\tilde{a} \to -\infty$, $\gamma_j > 0$ for all $j \geq 1$ in (15) but as $\tilde{a} \to \infty$ there is a single negative γ_j denoted $\gamma_0 \sim -\tilde{a}^2$
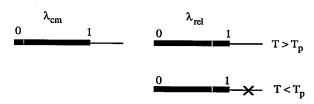

FIG. 1. Spectrum of kernel for center of mass ($\lambda_{\text{c.m.}}$) and relative motion (λ_{rel}) showing bound state (cross) for $T < T_p$. The continuum (heavy line) breaks up in λ_{rel} for $h > 0$.

(Ref. 14) which gives the dominant contribution to (15) of the OZ form $x^{-1/2}\exp(-2\tau x\{1-[(a-a_c)/a_c]^2\})$ for $a \gtrsim a_c$. The subdominant term is then the KW anomalous one $x^{-2}\exp(-2\tau x)$. For $a > a_c$, there is thus a mass gap which does not collapse as $h \rightarrow 0$. This is reminiscent of the correlation function of a uniaxial ferromagnet on going around the critical point in the (h,T) plane. Such a phenomenon might also supply a diagnostic test for this behavior experimentally.

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