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A Class of Interaction-Round-a-Face Models and Its Equivalence with an Ice-Type Model

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A new model (called the Temperley–Lieb interactions model) is introduced, in two-dimensional lattice statistics, on a square lattice $\mathscr{L}$. The Temperley–Lieb equivalence of this model to the six-vertex, self-dual Potts, critical hard-hexagons and critical nonintersecting string models is established. A graphical equivalence of this model to the six-vertex model generalizes this equivalence to noncritical cases of the above models. The order parameters of a specialization of this model are studied.

KEY WORDS: Six-vertex, Potts, hard-hexagons, and nonintersecting string models; "weak equivalence"; IRF model; magnetic hard squares; graphical equivalence.

1. INTRODUCTION

An interesting property of models in statistical mechanics is the "weak equivalence" between different models, where they have the same partition function per site. One such group of models is the six-vertex (6V), the self-dual Potts (SDP), the critical hard-hexagon (CHH), and the critical separable nonintersecting string (also called the NISI[(1)]) models. We may call this the "6V set" of models for convenience.

In 1971 Temperley and Lieb[(2)] showed the correspondence between the 6V and Potts models. Baxter[(3)] later expanded this equivalence to include the CHH model and postulated the inclusion of the NIS1 model into the 6V set. Perk and Schultz[(4)] confirmed this prediction. The method of Baxter and of Temperley and Lieb was to show that certain elementary matrices, whose product makes up the transfer matrices (the product of the

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transfer matrices is related to the partition function), lie in the same algebra for each model. This is known as the Temperley–Lieb algebra.

A graphical equivalence of the general Potts model and a staggered 6V model by Baxter *et al.*[5] illustrated the equivalence of the partition function for these models. It was no surprise then, when Perk and Wu[6] and also Truong[7] recently constructed explicit graphical equivalences of the separable $q^{1/2}$ state NIS model and the q-state Potts model in general. Perk and Wu[8] constructed the partition function of the critical NIS model on a square lattice using a graphical approach.

This leads one to look for other models that may be included in this equivalence family. This report announces one such model on a square lattice $\mathscr{L}$ called the Temperley–Lieb interactions (TLI) model. We define this model by Eqs. (3.1), subject to the restrictions (4.3), (4.5), and (4.8), and a set S described in Section 3. The CHH model and the magnetic hard-squares model on the line of essential singularities are specific cases of this model. In Section 3 we establish, under the restriction (3.3), its equivalence to the 6V set by ensuring that its local transfer matrices satisfy the Temperley–Lieb algebra. We then establish in general this equivalence graphically in Section 4.

The new model has a large number of disposable parameters: we suggest in Section 5 one particular specialization as a topic for further study. In particular, we use the corner transfer matrix method to obtain order parameters in Section 6.

2. TEMPERLEY–LIEB EQUIVALENCE

The following discussion relies on the discussion in Sections 2 and 3 of Baxter.[3] We use a prefix B to denote equations of that paper, e.g., (B2.1). The partition function of any model of the 6V set can be given as (B2.1) or (B2.26), depending on boundary conditions. These relations give Z as a product of local transfer matrices X_j that depend on local face, edge, or vertex conditions. A linearity property gives X_j the structure of (B2.16):

$$X_j = \rho(I + xU_j), \qquad j = 1,\ldots, n \tag{2.1}$$

where n is the number of faces or edges in a row of the lattice $\mathscr{L}$ and the U_j matrices satisfy (B2.19),

$$U_j^2 = q^{1/2} U_j \tag{2.2a}$$

$$U_j U_{j\pm 1} U_j = U_j \tag{2.2b}$$

$$U_i U_j = U_j U_i, \qquad |i-j| \geqslant 2 \tag{2.2c}$$

Equations (2.2) represent the Temperley–Lieb algebra relations. It is shown that since the set of matrices $\{U_j,\ j=1,\ldots,n\}$ form a finite-dimensional algebra, any two models with local transfer matrices satisfying (2.1) and (2.2), and with partition function given by (B2.26), will have equal partition functions. They must have the same values of ρ, x, and q, but since ρ is a trivial factor multiplying each local transfer matrix, the partition functions are simply related when the values of x and q correspond. We can then say that the models are equivalent.

Now, as in Section 3 of Ref. 3, the 6V set of models (except the SDP) can easily be given as "interaction-round-a-face" (IRF) models on a square lattice $\mathscr{L}$ turned through 45°. Then X_j represents the "addition" of a face and is given by (B3.2),

$$(X_j)_{\sigma\sigma'} = w(\sigma_j, \sigma_{j+1}, \sigma'_j, \sigma_{j-1}) \prod_{i\neq j} \delta(\sigma_i, \sigma'_i) \tag{2.3}$$

where σ and σ' are two spin sets $\{\sigma_1,\ldots,\sigma_n\}$ and $\{\sigma'_1,\ldots,\sigma'_n\}$.

Z is given by (B3.1) as well as (B2.26),

$$Z = \sum_{\sigma} \prod_{(ijkl)} w(\sigma_i, \sigma_j, \sigma_k, \sigma_l) \tag{2.4}$$

where the sum is over all spins $\sigma_1,\ldots,\sigma_N$ (N is the total number of sites in the lattice) and the product is over all N faces of the lattice. The spins can generally take on m values, so that the matrices X_j are of size $m^n \times m^n$.

As examples, the 6V model and the CHH model can be constructed as IRF models using the prescriptions (B3.12) and (B3.27):

For the 6V model:

$$(U_j)_{\sigma\sigma'} = \tfrac{1}{2}(1-\sigma_{j-1}\sigma_{j+1}) \exp[\tfrac{1}{2}\lambda\sigma_{j+1}(\sigma_j+\sigma'_j)] \prod_{i\neq j} \delta(\sigma_i, \sigma'_i) \tag{2.5a}$$

where $q = 4\cosh^2\lambda$ and each σ_j takes on value -1 and $+1$ ($m=2$).

For the CHH model:

$$(U_j)_{\sigma\sigma'} = q^{(-\sigma_j+\sigma'_j+\sigma_{j+1}+\sigma_{j-1})}\, \delta(\sigma_{j-1}, \sigma_{j+1}) \prod_{i\neq j} \delta(\sigma_i, \sigma'_i) \tag{2.5b}$$

where $q = \frac{1}{2}(3+\sqrt{5})$ and σ_j takes on the values 0 and 1 ($m=2$).

The spin set σ is restricted by the condition $\sigma_j\sigma_{j+1}=0$ for $j=1,\ldots,n-1$. Similarly for σ'. One can verify directly that U_j satisfy (2.2), and so form a Temperley–Lieb algebra.

3. THE MODEL

We define our model by giving $w(a, b, c, d)$. In general we shall choose

$$w(a, b, c, d) = f_{ab}\, g_{ad}\, \delta(a, c) + x h_{ba} k_{bc}\, \delta(b, d) \tag{3.1}$$

Figure 1 illustrates the interactions involved in this model. The diagonal bonds represent the delta functions and the f, g, h, and k functions are weights depending on the spins of the corners of the face separated by those bonds. The first argument of each function is the bonded spin value and the other is the unbonded spin value. This means that each function is associated with a triangular half-face of $\mathscr{L}$, rather than an edge. There is therefore no reason to require that the functions be symmetric (as one would expect of a function dependent on one edge only).

The spins a, b, c, d take the values $1,\ldots, m$ (or $0,\ldots, m-1$ as appropriate), where m is some given integer. In addition, throughout this paper we restrict all adjacent spin pairs to some given set S. Then S is the set of allowed spin pairs. For instance, in the CHH model (where $a, b, c, d = 0, 1$), we can take S to be the set

$$S = \{(0, 0), (0, 1), (1, 0)\} \tag{3.2}$$

i.e., a set of three elements, each being a pair of spins.

In general, the number of elements of S cannot exceed m^2. We impose the symmetry condition that if (a, b) belongs to S, then so does (b, a). The functions f_{ab}, g_{ab}, h_{ab}, k_{ab} are defined only for $(a, b) \in S$. [For the purpose of evaluating the partition function Z from (2.4) and (3.1) these restrictions are the same as taking f, g, h, k to be zero if their arguments do not belong to S, and then allowing each spin to independently take all m values. We find it preferable to restrict the definitions of f, g, h and k to spin pairs in S.]

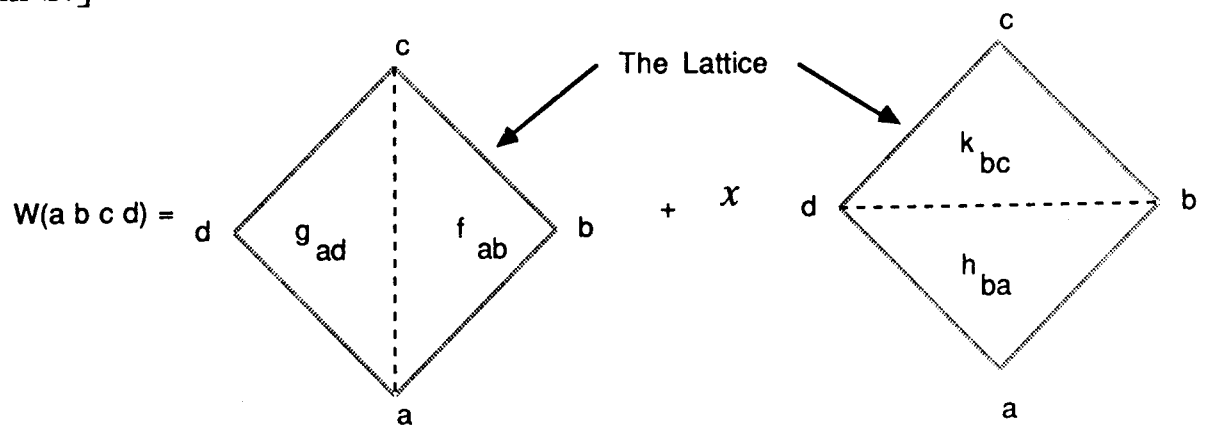

Fig. 1. A graphical representation of the terms in the face weights of the TLI model. The 'dashed' bonds make diagonally opposite spin values the same.

The matrix X_j is given by (2.3). To ensure that it has the form (2.1) (with $\rho = 1$), we take

$$f_{ab} = g_{ab} = 1, \qquad (a, b) \in S \tag{3.3}$$

(In the next section we shall relax this condition.) Then U_j is the matrix with entries

$$(U_j)_{\sigma\sigma'} = h_{\sigma_{j+1}\sigma_j} k_{\sigma_{j+1}\sigma'_j} \delta(\sigma_{j-1}, \sigma_{j+1}) \prod_{i \neq j} \delta(\sigma_i, \sigma'_i) \tag{3.4}$$

where σ and σ' are the spin sets $\{\sigma_1, ..., \sigma_n\}$ and $\{\sigma'_1, ..., \sigma'_n\}$. These spin sets are restricted by the condition $(\sigma_j, \sigma_{j+1}) \in S$ and $(\sigma'_j, \sigma'_{j+1}) \in S$. This implies that U_j, and so X_j, are generally matrices of size smaller than $m^n \times m^n$. Only when S contains all m^2 spin pairs does U_j take on the size $m^n \times m^n$.

If we define

$$V_{ab} = h_{ab} k_{ab}, \qquad (a, b) \in S \tag{3.5}$$

then it is shown in Appendix A that the Temperley–Lieb relations (2.2) are satisfied provided

$$V_{ab} V_{ba} = 1, \qquad (a, b) \in S \tag{3.6a}$$

$$\sum_{b \ni (a,b) \in S} V_{ab} = q^{1/2}, \qquad \forall a \tag{3.6b}$$

Surprisingly, this is quite a weak set of conditions: if S contains all m^2 spin pairs (a, b), then we have only $m(m+3)/2$ restrictions on the m^2 values of V_{ab}. (Note that since h_{ab} and k_{ab} need not be symmetric, neither need V_{ab}.) Thus, we have found a large class of models that are all equivalent to the 6V, Potts, and NIS models.

We have implicitly assumed that x is the same for all faces, which means that the corresponding 6V model is regular and the Potts model is self-dual. For $q \leqslant 4$ all the models are therefore critical (see Ref. 9). However, as will become clear in the next section, the equivalence remains true even if x is varied (for all the models) from face to face. Thus, it also links this model to the staggered 6V model and the non-self-dual, non-critical (and nonsolved) Potts model.

4. GRAPHICAL EQUIVALENCE WITH A 6V MODEL

In this section we demonstrate by graphical methods the equivalence of a site-dependent TLI model and an inhomogeneous 6V model and also a general q-state Potts model with anisotropic interactions. The self-dual case

of the Potts, and the homogeneous 6V models, are equivalent to the homogeneous TLI model of the last section as a special case of this graphical equivalence. The staggered separable $q^{1/2}$-state NIS model must then be equivalent to this expanded model.

We return to the expression (3.1) for $w(a, b, c, d)$ and no longer take f_{ab}, g_{ab} to be restricted by (3.3). We allow x (but not $f_{ab},..., k_{ab}$) to vary from face to face. Let x_i be its value on face i.

4.1. Graphical Expansion

From (3.1), each face function w is a sum of two terms:

$$w(a, b, c, d) = w_1(a, b, c, d) + w_2(a, b, c, d) \tag{4.1a}$$

where

$$w_1(a, b, c, d) = f_{ab}\, g_{ad}\, \delta(a, c) \tag{4.1b}$$

and

$$w_2(a, b, c, d) = x_i h_{ba} k_{bc}\, \delta(b, d) \tag{4.1c}$$

The N-fold product-over-faces in (2.4) can therefore be expanded as the sum of 2^N contributions each being a product of w_1 and w_2 terms.

We can associate with each term of the product expansion a bond graph G on the diagonals of the lattice $\mathscr{L}$. We do this by associating a bond appropriate to the delta function found in the term chosen from the weight w on each face. If we choose the first term containing $\delta(a, c)$, we link spins a and c together with a north–south (NS) bond. Similarly, for the second term, we link b with d by an east–west (EW) bond. See Fig. 1.

We then can write the partition function (2.4) as

$$Z_{\mathrm{TLI}} = \sum_G \sum_\sigma \prod_{\text{faces}} (w_1 \text{ or } w_2 \text{ factors}) \tag{4.2}$$

the outer sum being over all the 2^N bond-graphs, and each face contributing either a w_1 (for a NS bond) or w_2 (for an EW bond).

Within a graph G the sites of $\mathscr{L}$ are grouped into clusters (regarding isolated sites as one-component clusters). Thus, for instance, the sites A, B, C in Fig. 2 form a cluster. All the spins within a cluster must be equal (because the bonds correspond to Kronecker deltas).

4.2. Perimeter Polygons

Now we surround each cluster by a polygon. To do this we consider the dual lattice $\mathscr{L}_D$ of $\mathscr{L}$. Each site of $\mathscr{L}_D$ lies at the center of a face of $\mathscr{L}$,

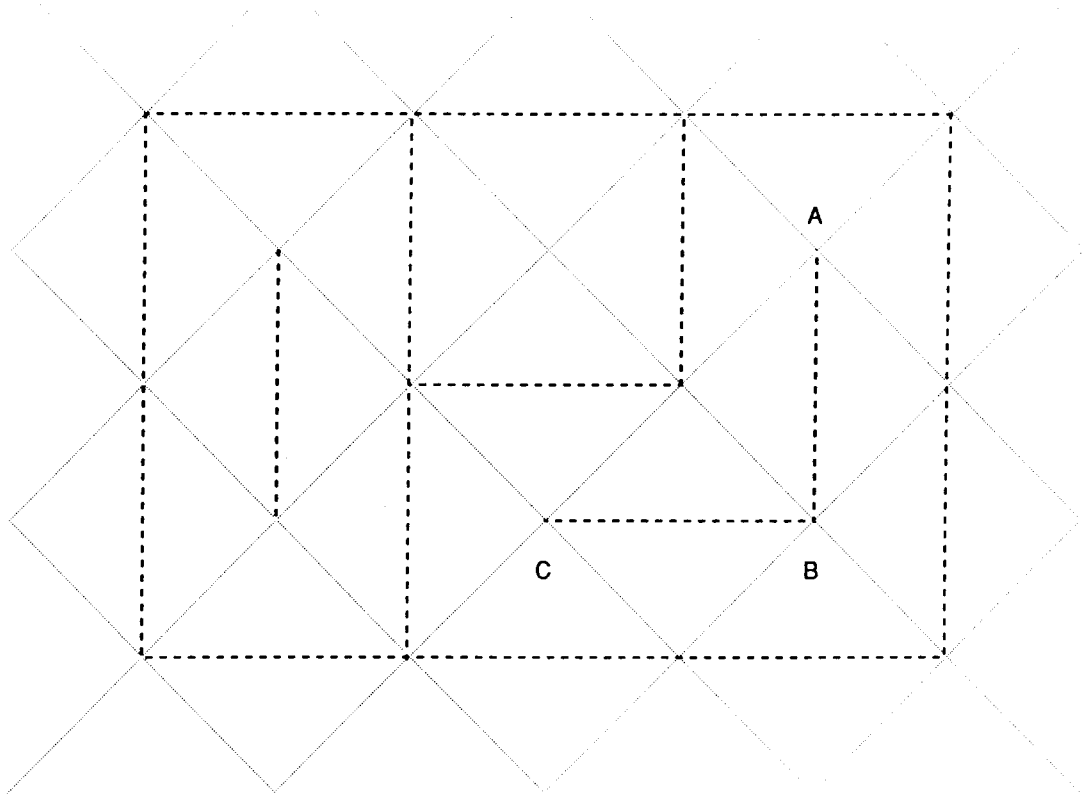

Fig. 2. A typical bond graph G showing the cluster A, B, C lying inside another cluster.

and hence on one of the bonds of G. We decompose $\mathscr{L}_D$ by regarding the bonds of G as "cutting" it, so as to separate edges on opposite sides of a bond as in Fig. 3.

The effect of this is to decompose $\mathscr{L}_D$ into disjoint polygons, as in Fig. 4.

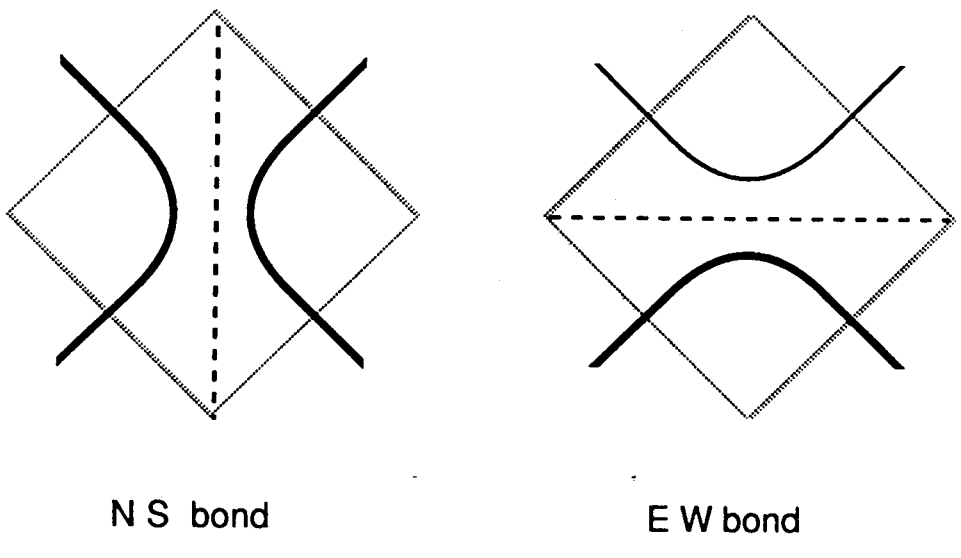

Fig. 3. The two types of bond on a face of the lattice. The dashed bonds separate corners of the lattice and the heavy lines are the associated polygon corners.

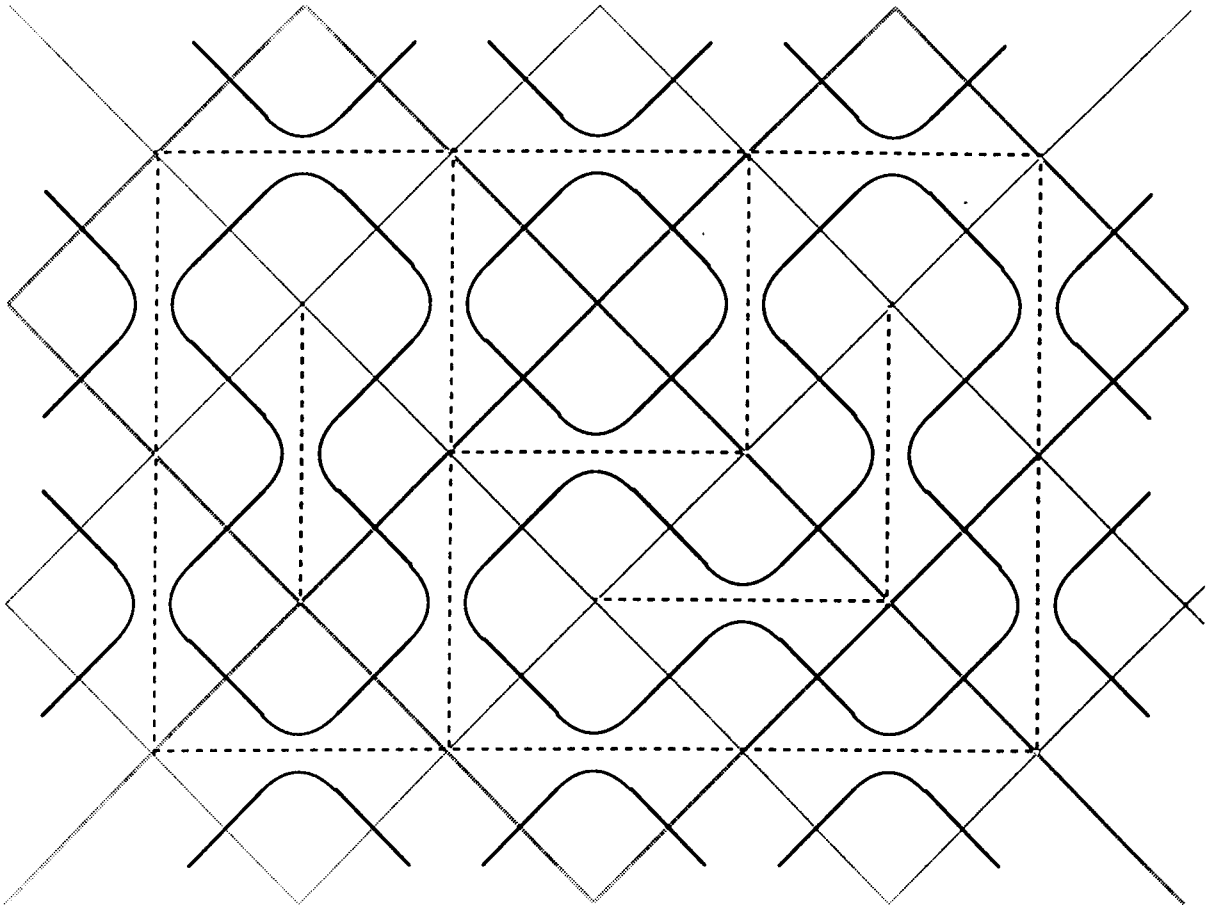

Fig. 4. The polygon decomposition of the bond graph of Fig. 2.

There is a one-to-one correspondence between graphs G and polygon decomposition of $\mathscr{L}_D$, so we can replace the outer sum in (4.2) by a sum over polygon decompositions.

We should allow for boundary effects. This can be done, but here we adopt the simpler viewpoint that $\mathscr{L}$ is arbitrarily large and G consists of an infinite cluster with finite clusters embedded inside it. (These finite clusters may in turn surround smaller ones.)

Then each polygon of $\mathscr{L}_D$ surrounds a finite cluster and is surrounded by another cluster (which may be the infinite one). Thus, with each polygon P we can associate an "interior" spin (that of the cluster immediately inside P) and an "exterior" spin (that of the cluster immediately outside P).

Now consider the w_1, w_2 factors in (4.2). We see that the w_1 term can be regarded as made up of the $\delta(a, c)$ factor (representing the NS bond), a factor f_{ab} associated with the right-hand polygon corner, and a factor g_{ad} associated with the left-hand polygon corner. Similarly, the w_2 term contains factors h_{ba}, k_{bc} associated with the lower and upper polygons, as well as an extra factor x_i (i being the label of the face). Thus, the spin-dependent

factor f_{ab}, g_{ab}, h_{ab}, and k_{ab} can all be associated with polygon corners as in Fig. 5.

Consider a complete polygon P and define

$$Q_{ab} = f_{ab}\, g_{ab}, \qquad R_{ab} = h_{ab} k_{ab} \tag{4.3}$$

Then in Appendix B it is shown that the product of the polygon corner weights is

$$Q_{ab} R_{ab} (Q_{ab} Q_{ba})^{n_7} (R_{ab} R_{ba})^{n_5} \tag{4.4}$$

where a is the exterior spin of P, and b the interior; n_7 and n_5 are non-negative integers determined by the shape and size of P. (The total number of corners is $4 + 4n_5 + 4n_7$.)

For the more specialized model discussed in Section 3 the products $Q_{ab}Q_{ba}$ and $R_{ab}R_{ba}$ are both one [in fact, for that case $Q_{ab} = 1$ and $V_{ab} = R_{ab}$ and by (3.6a), $R_{ab}R_{ba} = 1$], so (4.4) is independent of n_5 and n_7. Let us require that this property be true generally, i.e.,

$$Q_{ab} Q_{ba} = R_{ab} R_{ba} = 1 \qquad (a, b) \in S \tag{4.5}$$

Then the weight (4.4) is the same for all polygons, namely V_{ab}, where

$$V_{ab} = Q_{ab} R_{ab} \tag{4.6}$$

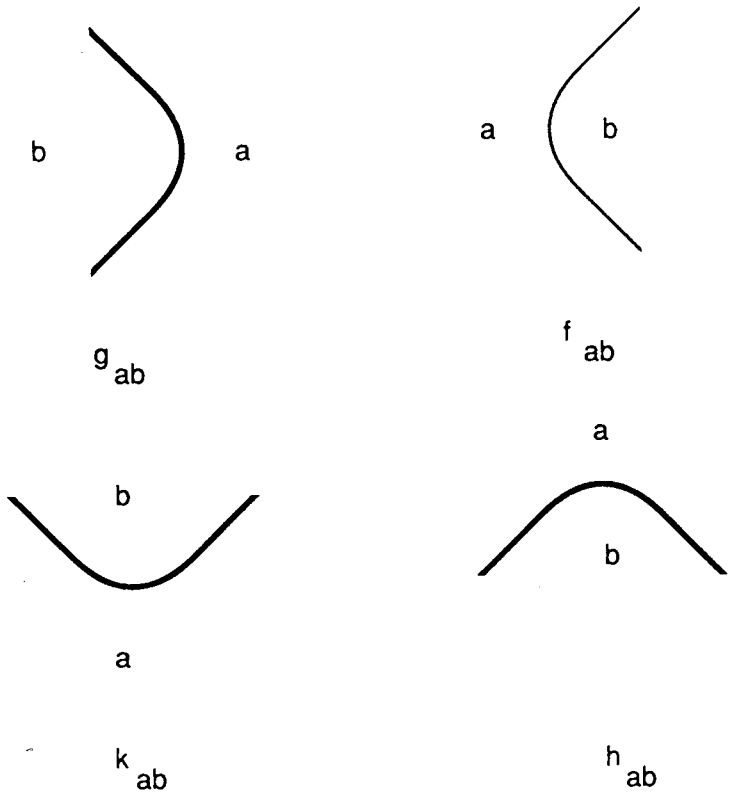

Fig. 5. The spin dependent functions with their associated polygon corners. The spins on either side of the polygons have values a and b.

Now these relations above give (4.2) as

$$Z = \sum_{\text{p.d.}} X \sum_{\sigma} \prod_{\text{polygons}} V_{ab} \tag{4.7}$$

where the outer sum is over all polygon decompositions of $\mathscr{L}_D$, X is the product of the x_i factors (one for each EW bond on G), the inner sum is over the values of the spins (one spin for each cluster), and the product is over all polygons of polygon weights V_{ab} (a and b are the exterior and interior spins of the polygons under consideration).

From (4.5), we have

$$V_{ab} V_{ba} = 1, \qquad (a, b) \in S \tag{4.8a}$$

so we still have the relation (3.6a). Let us suppose that (3.6b) also still applies, i.e.,

$$\sum_{b \ni (a,b) \in S} V_{ab} = q^{1/2}, \qquad \forall a \tag{4.8b}$$

A polygon may surround other polygons, which may surround other polygons, etc. This sequence must terminate with "elementary" polygons that contain no other polygons within them. Each contains a single cluster: the spin b of this cluster enters the σ-summand in (4.7) only via a single factor V_{ab}, where a is the spin exterior to the polygon. Thus we can sum at once over b. From (4.8), we get a factor $q^{1/2}$, which is *independent of a.*

Each elementary polygon therefore contributes a constant factor $q^{1/2}$ to (4.7). Apart from this, such polygons can be ignored. But then all the polygons that contained only elementary polygons within them become themselves "elementary." The same argument applies to them, and so on. Finally one is left only with the unweighted sum over the m values of the spin of the infinite cluster, giving a factor m. Hence (4.7) reduces to

$$Z = m \sum_{\text{p.d.}} X q^{p/2} \tag{4.9}$$

where p is the number of polygons in the decomposition of the square lattice $\mathscr{L}_D$.

4.3. Six-Vertex Model

Now consider a six-vertex model on $\mathscr{L}_D$,[10] where one places arows on the edges subject to the rule that at each site there be two arrows in and two out. Then there are six configurations as in Fig. 6.

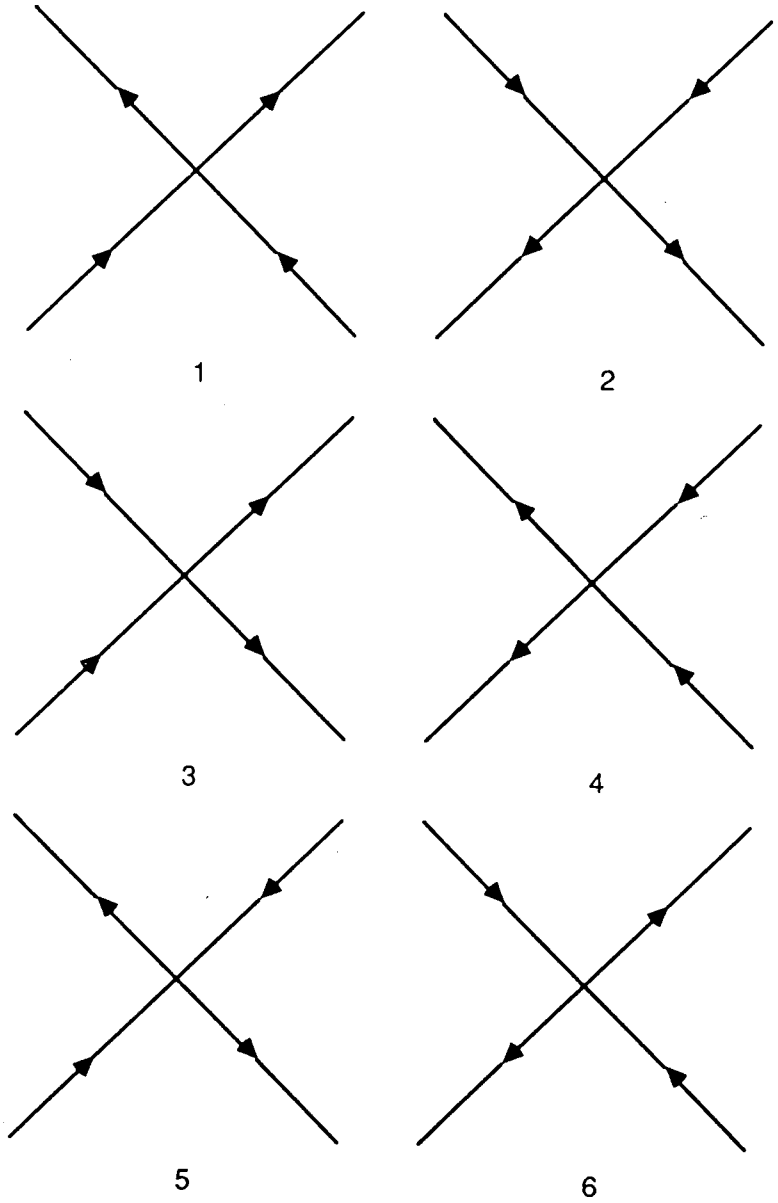

Fig. 6. The six arrow configurations of the 6V model on $\mathscr{L}_D$.

For site i (on face i on $\mathscr{L}$), let the six configurations (as labeled in Fig. 6) have weights

$$\omega_1,\ldots,\omega_6 = 1,\ 1,\ x_i,\ x_i,\ s^{-1}+x_i s,\ s+x_i s^{-1} \tag{4.10}$$

where s is some parameter. Then, by following the method of Refs. 5 and 10 we find that the partition function Z_{6V} of this model is given by[2]

$$Z_{6V} = \sum_{\text{p.d.}} X(s^2+s^{-2})^p \tag{4.11}$$

[2] It should be noted that our weights (4.10) differ from those in (12.4.2) of Ref. 10, in that we are using the same orientation of Fig. 6 for all sites of $\mathscr{L}_D$.

where again the sum is over polygon decompositions of $\mathscr{L}_D$, a factor x_i is associated with site i if the decomposition there is of the EW type (upper edges separated from lower), X is the product of these x_i factors, and p is the number of polygons.

Comparing (4.9) and (4.11) and choosing s to be related to q by

$$q^{1/2} = s^2 + s^{-2} \tag{4.12}$$

it follows that the partition functions of the TLI and six-vertex models are related by

$$Z_{\mathrm{TLI}} = mZ_{\mathrm{6V}} \tag{4.13}$$

4.4. Special Cases

While the equivalence (4.13) is true for all choices of the face parameters $x_1, \ldots, x_N$, we are usually only interested in a system with some translational invariance. The most interesting case to consider is when we divide the faces of $\mathscr{L}$ into two classes "black" and "white," as in a checkerboard (i.e., we divide $\mathscr{L}_D$ into two sublattices) and take

$$\begin{aligned} x_i &= x \quad \text{if } i \text{ is a black face} \\ x_i &= y \quad \text{if } i \text{ is a white face} \end{aligned} \tag{4.14}$$

(We note that the sites on $\mathscr{L}$ are naturally divided into two sublattices by the diagonal interactions.)

Then the product X in (4.7) and (4.9) is simply

$$X = x^{n_b} y^{n_w} \tag{4.15}$$

where n_b is the number of EW bonds of G on black faces and n_w is the number on white faces.

Then, by using the equivalence[3] between the six-vertex and Potts models,[5,10] we can establish that

$$Z_{\mathrm{TLI}} = my^{N/2} q^{-N/4} Z_{\mathrm{Potts}} \tag{4.16}$$

where Z_{Potts} is the partition function of the q-state Potts model on a square lattice of $N/2$ sites (actually one of the sublattices $\mathscr{L}$), with interaction coefficients K and L in the horizontal and vertical directions, where

$$x = (e^K - 1)\, q^{-1/2}, \qquad y = q^{1/2}(e^L - 1)^{-1} \tag{4.17}$$

[3] We can take the x_1 of Section 12.4 of Ref. 10 to be x and its x_2 to be $1/y$.

Explicitly,

$$Z_{\text{Potts}} = \sum_{s} \exp\left[K \sum_{\langle i,j \rangle} \delta(s_i, s_j) + L \sum_{\langle j,k \rangle} \delta(s_j, s_k) \right] \tag{4.18}$$

where the Potts spins $s_1, \ldots, s_{N/2}$ each take on the values $1, \ldots, q$ the $\langle i, j \rangle$ sum is over all horizontal edges of $\mathscr{L}_P$ (the Potts lattice), and the $\langle j, k \rangle$ sum over all vertical edges.

Particular interesting cases are as follows:

(i) $K, L \to 0$ ($x \to 0$, $y \to \infty$). This is the disordered limit of the Potts model ($T \to \infty$).

(ii) $K, L \to \infty$ ($x \to \infty$, $y \to 0$). This is the ordered limit ($T \to 0$), but we see that it differs from the disordered one only in the interchange of x and y, which is a manifestation of the duality of the Potts model.[11] In either limit the sum (4.2) is dominated by the graph G consisting of an infinite cluster linking all sites on one sublattice of $\mathscr{L}$, the other sites each being isolated "islands" within it. Thus, in these limits (and to any finite order in a perturbation about them), we are justified in considering only a single infinite cluster with finite clusters embedded inside it.

(iii) $(e^K - 1)(e^L - 1) = q$ ($x = y$). This is the self-dual case of the Potts model, corresponding to the homogeneous six-vertex and TLI models. For $q \leqslant 4$ the models are critical; for $q > 4$ the models are at a first-order transition point.[9]

5. A SPECIFIC EXAMPLE OF THE TLI MODEL

We now propose an interesting specific model of the TLI type. In choosing the model (or any specialization of the TLI model), the value of m and the set S of "possible" spin pairs are first chosen. We then have the corner weight functions at our disposal. (We will consider a homogeneous model.) Instead of completely specifying f_{ab}, g_{ab}, h_{ab}, and k_{ab}, we only need give V_{ab} satisfying (4.8).

Let us choose the spins to assume the values $0, \ldots, m-1$, where m is an integer. We choose S as

$$S = \{(ab) \ni ab = 0\} \tag{5.1}$$

so at least one of a and b must be zero. Then, by using (4.8) we find that

$$V_{00} = 1, \qquad V_{0b} = q^{-1/2}, \qquad V_{b0} = q^{1/2}, \qquad \forall b \neq 0 \tag{5.2}$$

and that

$$q^{1/2}(q^{1/2} - 1) = (m - 1) \tag{5.3}$$

Equations (5.1)–(5.3) define our specialized TLI model (STLI model). We note the freedom we still have in choosing the corner weight functions, as they must satisfy (4.3), (4.5), and (4.6), but are not explicitly given. Note that in this case V_{ab} and q are determined once the structure parameters m and S are given. (This is not true for more general choices of S.)

The model belongs to the homogeneous 6V set, so must be critical for $q \leqslant 4$ ($m \leqslant 3$), and be at a first-order transition for $q > 4$ ($m > 3$). The partition function per site $Z^{1/N}$ of the STLI model for $x > 0$ (the physical regime) is given by the six-vertex result[3] as

$$N^{-1} \ln Z = \ln\left(\frac{\sinh \lambda}{\sinh(\lambda - u)}\right) + 2 \sum_{n=1}^{\infty} e^{-2n\lambda} \frac{\sinh nu \sinh(\lambda - u)}{n \cosh n\lambda} \tag{5.4a}$$

for $m > 3$, where $x = \sinh u / \sinh(\lambda - u)$, $4 \cosh \lambda = 1 + (4m - 3)^{1/2}$ with $\lambda > 0$ and $0 < u < \lambda$, and

$$N^{-1} \ln Z = \ln\left(\frac{\sinh \mu}{\sinh(\mu - v)}\right) + \int_{-\infty}^{\infty} \frac{\cosh(\pi - 2\mu)t \sinh(\mu - v)t}{t \sinh \pi t \cosh \mu t} dt \tag{5.4b}$$

for $m < 3$, where $x = \sin v / \sin(\mu - v)$, $4 \cos \mu = 1 + (4m - 3)^{1/2}$ with $0 < \mu < \pi/2$, and $0 < v < \mu$. (We used $f_{ab} = g_{ab} = 1$ for these results.) In the next section we examine to order parameters for this model.

The CHH model is a special case, in which

$$m = 2, \qquad f_{ab} = g_{ab} = 1, \qquad h_{ab} = k_{ab} = q^{(a-b)/4}, \qquad q = \tfrac{1}{2}(3 + \sqrt{5}) \tag{5.5}$$

Also the STLI model with $m = 3$ ($q = 4$) (e.g., $f_{ab} = g_{ab} = 1$, $h_{ab} = k_{ab}$, $h_{0b} = 2^{-1/4}$, $h_{b0} = 2^{1/4}$, $h_{00} = 1$) can be seen to be the same as the magnetic hard-squares model[12] along its line of essential singularities.

6. ORDER IN THE STLI MODEL ($m \geqslant 3$)

We know that for $m > 3$ our homogeneous STLI model is at first-order transition point. This implies that there exists broken symmetry within the system and hence order. We will now investigate this order.

In Section 4, when discussing the equivalence of any TLI model to a six-vertex model, we fixed the boundary effects by considering any bond graph G to consist of an infinite boundary cluster with finite clusters embedded in it. We now explore the correlation between the value of this boundary spin and a spin deep within the lattice.

Let f_{ab} be the probability that a spin σ_0 deep in the lattice is equal to a, given that the boundary spin σ_∞ is equal to b. We define the function $F(\sigma_0, \sigma_\infty)$ as

$$F(\sigma_0, \sigma_\infty) = \delta(\sigma_0, a)\, \delta(\sigma_\infty, b) \tag{6.1}$$

One immediately can see that

$$f_{ab} = \langle F(\sigma_0, \sigma_\infty) \rangle \tag{6.2}$$

and so from the formulation of the TLI model as (2.4) we have

$$f_{ab} = \frac{\sum_\sigma \delta(\sigma_0, a)\, \delta(\sigma_\infty, b)\ (\text{face weights})}{\sum_\sigma \delta(\sigma_0, b)\ (\text{face weights})} \tag{6.3}$$

where the sum is over all spins on the lattice $\mathscr{L}$. By using the polygon decomposition formulation (4.7) of the TLI model, we have

$$f_{ab} = \frac{\sum_{\text{p.d.}} X \sum_{\sigma'} \delta(\sigma_0, a) \prod_{\text{polygons}} V_{cd}}{\sum_{\text{p.d.}} X \sum_{\sigma'} \prod_{\text{polygons}} V_{cd}} \tag{6.4}$$

where, as before, the outer sum is over polygon decompositions of $\mathscr{L}_D$, X is the product of x factor, the inner sum is over all spins except those in the boundary cluster, and the product is over all polygons of polygon weights V_{cd} (c and d are the exterior and interior spin values of the polygons under consideration). We can sum out all polygons except those surrounding σ_0, as in Section 4.2, to obtain

$$f_{ab} = \frac{\sum_{l=0}^{\infty} \sum_{\text{p.d.}}^{*} X q^{(p-l)/2} (V^l)_{ba}}{\sum_{\text{p.d.}} X q^{p/2}} \tag{6.5}$$

where l is the number of polygons surrounding σ_0 and the starred sum is restricted to those decompositions containing l polygons round σ_0.

The matrix V can be diagonalized. Let $\{\lambda_\gamma : \gamma = 0, 1, \ldots, m-1\}$ be the complete set of eigenvalues of V, and let Y be the matrix that diagonalizes V; thus,

$$Y^{-1} V Y = V_d \tag{6.6}$$

where $(V_d)_{\alpha\beta} = \lambda_\alpha\, \delta(\alpha, \beta)$. We can therefore show that

$$f_{ab} = \sum_{\gamma=0}^{m-1} Y_{b\gamma}\, Y_{\gamma a}^{-1}\, G(\lambda_\gamma^2/q) \tag{6.7a}$$

where

$$G(z) = \sum_{l=0}^{\infty} z^{l/2} \mathscr{U}_l \tag{6.7b}$$

and defining $\mathcal{U}_l$ as

$$\mathcal{U}_l = \sum_{\text{p.d.}}^{*} Xq^{p/2} \Big/ \sum_{\text{p.d.}} Xq^{p/2} \tag{6.7c}$$

where the sum in (6.7b) runs through all even values of l when σ_0 and σ_∞ lie on the same sublattice, and runs through all odd values of l when σ_0 and σ_∞ lie on different sublattices. This qualification of the sum (6.7b) results from the fact that if σ_0 and σ_∞ lie on the same sublattice, only an even number of polygons can lie between them. Similarly, if σ_0 and σ_∞ lie on different sublattices, only an odd number of polygons can between them.

The task left is to calculate the $\mathcal{U}_l$, since Y and λ_γ are known (V is known). It can be seen from (6.7c) that $\mathcal{U}_l$ is the probability that l polygons on $\mathcal{L}_D$ surround a site σ_0 deep in the lattice. We know that our STLI model is equivalent to a homogeneous 6V model on the lattice $\mathcal{L}_D$ by placing arrows on the polygons in the appropriate manner. We also know that the homogeneous 6V model is equivalent to the dichromatic formulation of the self-dual Potts model. In another paper[13] we introduce generalized percolation probabilities P_n and P'_n and show that P_n is the probability that $2n$ polygons surround a central site on the Potts lattice (which is a sublattice of $\mathcal{L}$). Similarly, P'_n is the probability that $2n-1$ polygons surround a central site. We can therefore identify

$$\mathcal{U}_{2n} = P_n \qquad \text{and} \qquad \mathcal{U}_{2n-1} = P'_n \tag{6.8}$$

In Ref. 13 we calculate P_n and P'_n using the corner transfer matrix method; more precisely we find generating functions for P_n and P'_n. We can immediately deduce

$$G(z) = \sum_{n=0}^{\infty} z^n P_n = \prod_{j=1}^{\infty} \frac{\{1 + x^{4j-2}(qz-2) + x^{8j-4}\}}{\{1 + x^{4j-2}(q-2) + x^{8j-4}\}} \tag{6.9a}$$

when σ_0 and σ_∞ lie on the same sublattice, and

$$G(z) = \sum_{n=1}^{\infty} z^{n-1/2} P'_n = z^{1/2} \prod_{j=1}^{\infty} \frac{\{1 + x^{4j}(qz-2) + x^{8j}\}}{\{1 + x^{4j}(q-2) + x^{8j}\}} \tag{6.9b}$$

when σ_0 and σ_∞ lie on different sublattices, where $x + x^{-1} = q-2$, $0 < x < 1$, and $q = [2m - 1 + (4m-3)^{1/2}]/2$.

We can now find f_{ab}. When σ_0 is deep inside the lattice, f_{ab} is independent of the position of σ_0, except insofar it depends on the sublattice. Using the eigenvalues of V, calculated as $q^{1/2}$, $1-q^{1/2}$, and 0 (0 is $m-2$ times degenerate), we have

$$
\begin{aligned}
f_{00} &= \frac{q+(m-1)\,G(z_1)}{q+m-1} \\
f_{d0} &= \frac{1-G(z_1)}{q+m-1} \\
f_{0b} &= \frac{q[1-G(z_1)]}{q+m-1} \\
f_{bb} &= \frac{(m-1)+qG(z_1)+(m-2)(q+m-1)\,G(0)}{(m-1)(q+m-1)} \\
f_{ab} &= \frac{(m-1)+qG(z_1)-(q+m-1)\,G(0)}{(m-1)(q+m-1)}
\end{aligned}
\tag{6.10}
$$

where $1 \leqslant a \leqslant m-1$, $1 \leqslant b \leqslant m-1$, $a \neq b$, and $z_1 = (q^{-1/2}-1)^2$ with $q = [2m-1+(4m-3)^{1/2}]/2$. From now on we distinguish between the two sublattice positions of σ_0 relative to σ_∞. When σ_0 and σ_∞ lie on different sublattices we use f'_{ab} to denote f_{ab} and $G'(z)$ to denote $G(z)$. Otherwise, we continue to use f_{ab} and $G(z)$. Then, f'_{ab} is obtained from (6.10) by replacing $G(z)$ with $G'(z)$ [note $G'(0)=0$].

From f_{ab} and f'_{ab} various measures of the order of the system are available. For a disordered system we expect the probability (6.2) to be independent of the position σ_0 relative to σ_∞. We also expect there to be no correlation between the values of σ_0 and σ_∞ and the states 1,..., $m-1$ to be equivalent. Thus, in this case there should exist a parameter u such that

$$
\begin{aligned}
f_{ab} = f_{ab} &= 1-(m-1)u \qquad \text{if} \quad a=0 \\
&= u \qquad\qquad\quad \text{if} \quad a>0 \quad \text{and} \quad \forall b
\end{aligned}
\tag{6.11}
$$

So if f_{ab} and f'_{ab} do not satisfy the conditions (6.11), the model is in a state of some order. For (6.11) to hold we must have

$$
G(z_1) = G(0) = G'(z_1) = 0 \tag{6.12}
$$

giving

$$
u = (q+m-1)^{-1}
$$

We can therefore use (6.12) as a test for disorder and use $G(z_1)$, $G(0)$, and $-G'(z_1)$ [as $G'(z_1)$ is negative] as order parameters instead of f_{ab} and f'_{ab}.

From (6.9) we deduce that $G(z_1)$, $G(0)$, and $-G'(z_1)$ are monotonically increasing functions of m, obeying

$$
0 \leqslant G(z_1),\, G(0),\, -G'(z_1) < 1 \tag{6.13}
$$

and are all zero at $m=3$. So the $m=3$ STLI model is disordered with

$$u = 1/6 \tag{6.14}$$

For $m>3$ the models are ordered to some extent, with the values of $G(z_1)$, $G(0)$, and $-G'(z_1)$ all providing a measure of this order. At $m=\infty$ these functions achieve their maximum values of 1, and so maximum order exists where

$$f_{ab} = \delta(a, b) \qquad \text{and} \qquad f'_{ab} = \lim_{q \to \infty} q^{-1/2} V_{ba} \tag{6.15}$$

for $a, b = 0,..., m-1$; [V_{ba} defined by (5.2)]. This reflects the fact that for m large all the spins on one sublattice (the one containing the boundary spins) are equal to σ_∞, while the other spins are independent, each spin σ_0 having a weight proportional to $V_{\sigma_\infty, \sigma_0}$. Figure 7 displays all three functions over a range of values of m.

The STLI model has been defined for integer values of m and the order of the system has been discussed for $m \geqslant 3$. However, the functions $G(z_1)$, $G(0)$, and $-G'(z_1)$ all can be defined for real $m \geqslant 3$. We are now able to examine the behavior of the orders as m approaches 3 from above. The order functions behave as

$$\begin{aligned} G(z_1) &\sim \exp\{-\pi^2/36[(m-3)/3]^{1/2}\} \\ G(0) &\sim 2\exp\{-\pi^2/16[(m-3)/3]^{1/2}\} \\ -G'(z_1) &\sim \exp\{-\pi^2/36[(m-3)/3]^{1/2}\} \end{aligned} \tag{6.16}$$

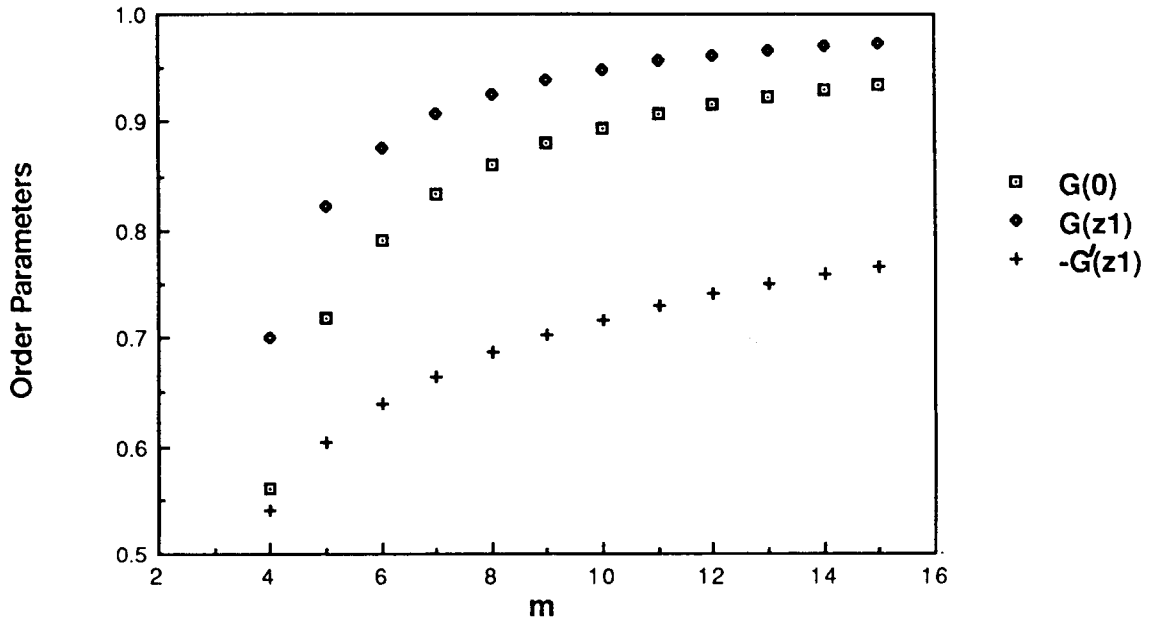

Fig. 7. Order parameters $G(0)$, $G(z_1)$, $-G'(z_1)$ over a range of m. (All three functions are 0 at $m=3$.)

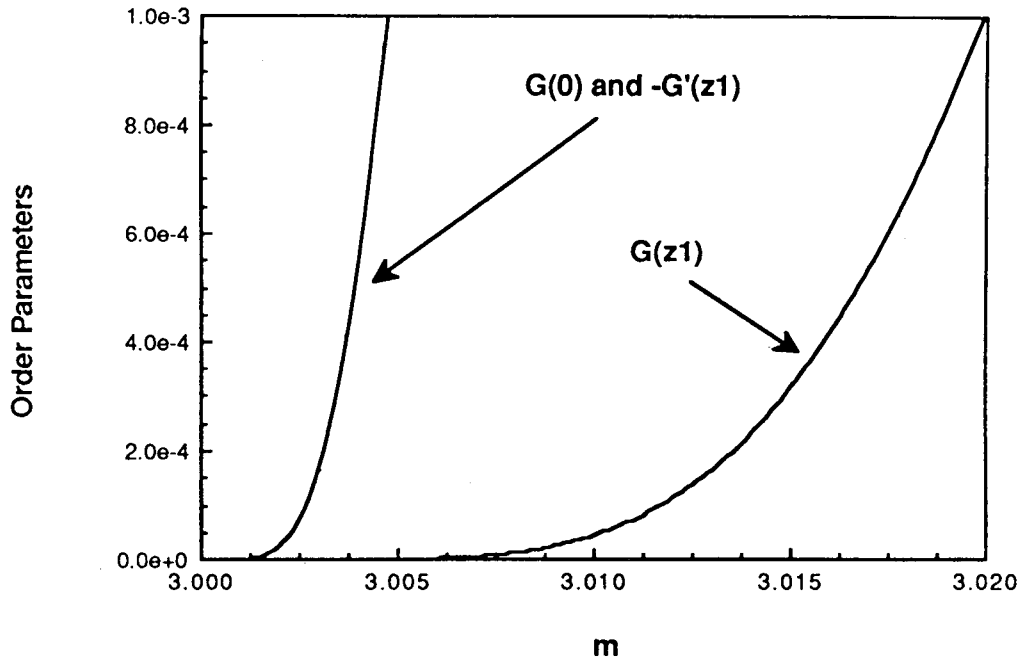

Fig. 8. The behavior of the order parameters near $m = 3$. ($G(0)$ and $G(z_1)$ lie on the same curve within the resolution of this graph.)

as m approaches 3 from above ($q \to 4^+$). [Note that $G(z)$ depends on x, q, and z, where x and q are functions of m and if $z = z_1$, it, too, is a function of m.] Figure 8 shows these functions close to $m = 3$. We note that the asymptotic equations (6.16) are reasonably good approximations only for $m \lesssim 3.01$ and so this behavior does not show up in the values of the order functions at integer values of m.

We have a description of the order in the system via (6.1), (6.2), (6.9), and (6.10) for any STLI model with $m \geqslant 3$. The model with $m = 3$ is disordered; at values of $m > 3$ there is some order, with greater order in models with greater m; and the model with $m = \infty$ has maximum order. We also have obtained asymptotic expressions (6.16) for the behavior of this order for real m close to 3.

APPENDIX A

We show in this appendix that conditions (3.6a), (3.6b) are required to hold if U_j of Eq. (3.4) is to satisfy the Temperley–Lieb relations (2.2). First we impose (2.2a), i.e., $U_j^2 = q^{1/2} U_j$, $j = 1, \ldots, n$:

$$(U_j)_{\sigma\sigma'}(U_j)_{\sigma'\sigma''} = \sum_{\sigma'} h_{\sigma_{j+1}\sigma_j} k_{\sigma_{j+1}\sigma_j'} \delta(\sigma_{j-1}, \sigma_{j+1})\, h_{\sigma_{j+1}'\sigma_j'} k_{\sigma_{j+1}'\sigma_j''} \delta(\sigma_{j-1}', \sigma_{j+1}')$$

$$\times \prod_{i \neq j} \delta(\sigma_i, \sigma_i') \prod_{i \neq j} \delta(\sigma_i', \sigma_i'')$$

We sum over σ_i, $i \neq j$:

$$= \sum_{\sigma_j'} h_{\sigma_{j+1}\sigma_j} k_{\sigma_{j+1}\sigma_j'} h_{\sigma_{j+1}\sigma_j'} k_{\sigma_{j+1}\sigma_j''} \delta(\sigma_{j-1}, \sigma_{j+1}) \prod_{i \neq j} \delta(\sigma_i, \sigma_i'')$$

$$= (U_j)_{\sigma\sigma'} \sum_{\sigma_j'} h_{\sigma_{j+1}\sigma_j'} k_{\sigma_{j+1}\sigma_j'}$$

which must equal $q^{1/2}(U_j)_{\sigma\sigma''}$ if (2.2a) is to be satisfied. Therefore

$$\sum_{b \ni (a,b) \in S} h_{ab} k_{ab} = q^{1/2} \qquad \forall a$$

As h_{ab} and k_{ab} are only defined for $(a, b) \in S$, the sum is over b values such that $(a, b) \in S$ [this is implied by the fact that $(\sigma_{j+1}, \sigma_j')$ is on an edge of the lattice].

Second, we impose (2.2b), i.e., $U_j U_{j\pm 1} U_j = U_j$, $j = 1, \ldots, n$:

$$(U_j)_{\sigma\sigma'}(U_{j+1})_{\sigma'\sigma''}(U_j)_{\sigma''\sigma'''}$$

$$= \sum_{\sigma'} \sum_{\sigma''} h_{\sigma_{j+1}\sigma_j} k_{\sigma_{j+1}\sigma_j'} h_{\sigma_{j+2}'\sigma_{j+1}'} k_{\sigma_{j+2}'\sigma_{j+1}''} h_{\sigma_{j+1}''\sigma_j''} k_{\sigma_{j+1}''\sigma_j'''}$$
$$\times \delta(\sigma_{j-1}, \sigma_{j+1}) \delta(\sigma_j', \sigma_{j+2}') \delta(\sigma_{j-1}'', \sigma_{j+1}'')$$
$$\times \prod_{i \neq j} \delta(\sigma_i, \sigma_i') \prod_{i \neq j+1} \delta(\sigma_i', \sigma_i'') \prod_{i \neq j} \delta(\sigma_i'', \sigma_i''')$$

We sum over σ_i', $i \neq j, j+1$ and σ_i'', $i \neq j, j+1$:

$$= \sum_{\sigma_j'} \sum_{\sigma_{j+1}'} \sum_{\sigma_j''} \sum_{\sigma_{j+2}''} h_{\sigma_{j+1}\sigma_j} k_{\sigma_{j+1}\sigma_j'} h_{\sigma_{j+2}\sigma_{j+1}'} k_{\sigma_{j+2}\sigma_{j+1}''} h_{\sigma_{j+1}''\sigma_j''} k_{\sigma_{j+1}''\sigma_j'''}$$
$$\times \delta(\sigma_{j-1}, \sigma_{j+1}) \delta(\sigma_j', \sigma_{j+2}) \delta(\sigma_{j-1}, \sigma_{j+1}'')$$
$$\times \delta(\sigma_{j+1}, \sigma_{j+1}') \delta(\sigma_j', \sigma_j'') \delta(\sigma_{j+1}'', \sigma_{j+1}''') \prod_{i \neq j, j+1} \delta(\sigma_i, \sigma_i''')$$

We sum over σ_{j+1}'', σ_{j+1}', σ_j'', σ_j' in that order, replacing σ_{j-1} with σ_{j+1} via $\delta(\sigma_{j-1}, \sigma_{j+1})$ wherever it occurs:

$$= h_{\sigma_{j+1}\sigma_j} k_{\sigma_{j+1}\sigma_{j+2}} h_{\sigma_{j+2}\sigma_{j+1}} k_{\sigma_{j+2}\sigma_{j+1}} h_{\sigma_{j+1}\sigma_{j+2}} k_{\sigma_{j+1}\sigma_j'''}$$
$$\times \delta(\sigma_{j-1}, \sigma_{j+1}) \prod_{i \neq j} \delta(\sigma_i, \sigma_i')$$

$$= (U_j)_{\sigma\sigma'} h_{\sigma_{j+1}\sigma_{j+2}} k_{\sigma_{j+1}\sigma_{j+2}} h_{\sigma_{j+2}\sigma_{j+1}} k_{\sigma_{j+2}\sigma_{j+1}}$$

which must equal $(U_j)_{\sigma\sigma'}$ if (2.2b) is to be satisfied. Therefore

$$h_{ab} k_{ab} h_{ba} k_{ba} = 1 \qquad \text{for} \quad (a, b) \in S$$

since $(\sigma_{j+1}, \sigma_{j+2})$ is an edge of the lattice.

It is easy to see that using $U_j U_{j-1} U_j = U_j$ provides a similar result, as the summing procedure does not change. In fact

$$h_{\sigma_{j-1}\sigma_j} k_{\sigma_{j-1}\sigma_j} h_{\sigma_j\sigma_{j-1}} k_{\sigma_j\sigma_{j-1}} = 1 \qquad \text{for} \quad (\sigma_j, \sigma_{j-1}) \in S$$

is the result produced. So only one restriction on $h_{ab}k_{ab}$ is produced from $U_j U_{j\pm 1} U_j$. Finally, (2.2c) is automatically satisfied:

$$(U_j)_{\sigma\sigma'}(U_k)_{\sigma'\sigma''} = \sum_{\sigma'} h_{\sigma_{j+1}\sigma_j} k_{\sigma_{j+1}\sigma'_j} h_{\sigma'_{k+1}\sigma'_k} k_{\sigma'_{k+1}\sigma''_k}$$
$$\times \delta(\sigma_{j-1}, \sigma_{j+1})\, \delta(\sigma'_{k-1}, \sigma'_{k+1}) \prod_{i\neq j} \delta(\sigma_i, \sigma'_i) \prod_{l\neq k} \delta(\sigma'_l, \sigma''_l)$$

Now $|j-k| \geqslant 2$ implies $j \neq k \pm 1$ and $j \neq k$. Let us separate the sums over σ'_j and σ'_k and sum over these spins:

$$= \sum_{\{\sigma'_i, i\neq j,k\}} h_{\sigma_{j+1}\sigma_j} k_{\sigma'_{j+1}\sigma''_j} h_{\sigma_{k+1}\sigma_k} k_{\sigma'_{k+1}\sigma''_k}$$
$$\times \delta(\sigma_{j-1}, \sigma_{j+1})\, \delta(\sigma_{k-1}, \sigma_{k+1}) \prod_{i\neq j,k} \delta(\sigma_i, \sigma''_i)$$

But this is symmetrical in j and k, and so equals $(U_k)_{\sigma\sigma'}(U_j)_{\sigma'\sigma''}$.

APPENDIX B

In this appendix we show that the product of corner weights around any complete polygon in a polygon decomposition of $\mathscr{L}_D$ is given by (4.4).

Let us consider one polygon. The corner can "point" in four directions, with the inside of the polygon being on either side of the corner in each case. This gives eight types of polygon corners. For convenience let us consider arrows being placed on the edges of the polygon so as to follow around the perimeter in one direction. We will choose the anticlockwise direction, but note that these arrows are only an analytical help and can be chosen either way. The inside of the polygon is then always to the left of the direction of an arrow on any edge. To each corner we associate the corresponding corner weights f_{ab}, g_{ab}, h_{ab}, and k_{ab}.

Let n_1, n_2, n_3, n_4 (n_5, n_6, n_7, n_8) be the respective numbers of left-turn (right-turn) corners with the corner pointing in the S, N, E, W directions. This section details relations among the n_j. The list is suffcient to obtain all possible relations among the n_j, as will be explained.

An observer going round the polygon in the direction of the arows must turn left four more times than right. This gives the relation

$$n_1 + n_2 + n_3 + n_4 = n_5 + n_6 + n_7 + n_8 + 4 \tag{B.1}$$

An observer moving in a vertical line up the lattice $\mathscr{L}_D$ must leave the interior of P as often as he enters it. So

$$n_2 + n_5 = n_1 + n_6 \tag{B.2}$$

Similarly, in the horizontal direction we get

$$n_3 + n_8 = n_4 + n_7 \tag{B.3}$$

The number of sources of NW-pointing arrows is the same as the number of sinks, as each arrow must have a beginning and an end. Therefore

$$n_3 + n_5 = n_2 + n_8 \tag{B.4}$$

Similarly, for NE-pointing arrows

$$n_3 + n_6 = n_1 + n_8 \tag{B.5}$$

and for SW pointing arrows

$$n_4 + n_5 = n_2 + n_7 \tag{B.6}$$

and for SE arrows

$$n_1 + n_7 = n_4 + n_6 \tag{B.7}$$

We now simplify these relations by combining them. Relations (B.2), (B.4), and (B.5) combine to give

$$n_1 = n_2, \qquad n_5 = n_6 \tag{B.8}$$

and relations (B.3), (B.4), and (B.6) combine to give

$$n_3 = n_4, \qquad n_7 = n_8 \tag{B.9}$$

Combining (B.1), (B.4), (B.8), and (B.9) results in the relations

$$n_1 = n_5 + 1 \tag{B.10}$$

and

$$n_3 = n_7 + 1 \tag{B.11}$$

We can use these to express $n_1, \ldots, n_8$ in terms of n_5 and n_7:

$$\begin{aligned} n_1 &= n_2 = n_5 + 1 \\ n_3 &= n_4 = n_7 + 1 \\ n_6 &= n_5 \\ n_8 &= n_7 \end{aligned} \tag{B.12}$$

and (B.1)–(B.7) are now all satisfied.

By considering polygons around a rectangular cluster we can find examples for which n_5 and n_7 take any pair of nonnegative integer values. This implies that there are no further independent relations among $n_1, \ldots, n_8$.

We now form the product of all the corner weights for the polygon

$$\prod_c (\text{weights}) = k_{ab}^{n_1} h_{ab}^{n_2} g_{ab}^{n_3} f_{ab}^{n_4} k_{ba}^{n_5} h_{ba}^{n_6} g_{ba}^{n_7} f_{ba}^{n_8} \tag{B.13}$$

where the product is over all corners of the polygon.

Using (B.12), we find

$$\prod_c (\text{weights}) = (k_{ab} h_{ab})(g_{ab} f_{ab})(k_{ab} h_{ab} k_{ba} h_{ba})^{n_5} (g_{ab} f_{ab} g_{ba} f_{ba})^{n_7} \tag{B.14}$$

Now applying the definition (4.3), (B.14) becomes

$$\prod_c (\text{weights}) = Q_{ab} R_{ab} (Q_{ab} Q_{ba})^{n_7} (R_{ab} R_{ba})^{n_5} \tag{B.15}$$

This relation (B.15) is the equation (4.4) that was presented. Note that as n_5 and n_7 are independent, no less specific relations than (4.5) can be provided to obtain (4.7). Also, the total number of corners is

$$\sum_j n_j = 4 + 4n_5 + 4n_7$$

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